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A Mathematical Model for Compliant Risers

MIT Sea Grant
College Program

Massachusetts
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A MATHEMATICAL MODEL FOR COMPLIANT RISERS

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ABSTRACT

The objective of this work is to present a general non-linear mathematical model describing the global behavior of a compliant riser idealized as a slender, non-rotationally uniform rod with bending, extensional and torsional degrees of freedom. This model includes the effects of external and internal pressure and speed of the internal fluid on the system.

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RELATED SEA GRANT REPORTS

1. Non-linear Statics of Non-Rotationally Uniform Rods with Torsion, by N. M. Patrikalakis and C. Chryssostomidis, MIT SG Report No. 85-18, 1985.
2. Linear Dynamics of Compliant Risers, by N. M. Patrikalakis and C. Chryssostomidis, MIT SG Report No. 85-19, 1985.

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NOMENCLATURE

$\vec{a}$	absolute acceleration of C, $\vec{a}=[a^{\zeta},a^{\xi},a^{\eta}]\cdot U''$
$A_i,A_o;A_b$	total inner and outer cross-sectional area of riser tubes; total outer cross-sectional area of riser tubes and buoyancy modules
$\vec{b}$	unit vector in binormal direction
B_b	buoyancy per unit length of buoyancy modules in water
B^*	$\rho_w g A_o$ for $y \leq h_w$ and zero otherwise
c	mean internal fluid speed; for our application $\rho_i c^2 \ll p$
C	rotation matrix, see equation (II.14); or centroid of a cross-section
D	maximum dimension of a cross-section
e	strain
E	Young's modulus
EA	extensional rigidity
EI^{bb},EI^{nb}	bending rigidity of a cross-section around $\vec{b}$; cross-product of bending rigidity around $\vec{n}$ and $\vec{b}$
EI^{xx},EI^{yy},EI^{xy}	bending rigidities of a cross-section around x, y and the cross-product of bending rigidity around x and y
$EI^{\xi\xi},EI^{\eta\eta}$	maximum and minimum bending rigidities of a cross-section
$EI^{\xi\eta}$	cross-product of bending rigidity around $\vec{\xi}$ and $\vec{\eta}$
$EI_e^{\xi\xi};EI_e^{\eta\eta}$	effective rigidities $EI^{\xi\xi}-c^2J_i^{\xi\xi};EI^{\eta\eta}-c^2J_i^{\eta\eta}$
f	orientation angle, see Figure B.1

NOMENCLATURE (continued)

$\vec{F}', \vec{F}$	internal forces, $\vec{F}'=[T',Q^{\xi},Q^{\eta}]\cdot U''$; for $\vec{F}$ see equation (II.92)
$\vec{F}_H$	external hydrodynamic force per unit length (excluding gravity effects); $\vec{F}_H=[F_H^{\zeta},F_H^{\xi},F_H^{\eta}]\cdot U''$
$\vec{F}_i$	force per unit length due to internal flow (excluding gravity effects)
g	acceleration of gravity
GI^P, GI_e^P	torsional and effective torsional rigidity, $GI^P-c^2J_i^{\zeta\zeta}$
h_i, h_w	internal fluid and salt water elevations above the axes origin
$\vec{H}_C$	$\vec{H}_{R,C} + \vec{H}_{i,C}$
$\vec{H}_{i,C};\vec{H}_{R,C}$	angular momentum per unit length about C of internal fluid; riser materials and buoyancy modules
$J_R;J_i$	mass inertia per unit length tensor of riser material and buoyancy modules: $J_R=\text{diag}[J_R^{\zeta\zeta},J_R^{\xi\xi},J_R^{\eta\eta}]$; and internal fluid $J_i=\text{diag}[J_i^{\zeta\zeta},J_i^{\xi\xi},J_i^{\eta\eta}]$, where diag[.] stands for diagonal matrix
ℓ_1,ℓ_2,ℓ_3	direction cosines of $\vec{b}$ with respect to $C\vec{\zeta}\vec{\xi}\vec{\eta}$
L, L_b	unstretched riser length, buoyancy module length
m^{ζ}, m	$(W_R+W_b)/g, (W_R+W_i+W_b)/g$
$\vec{M}$	restoring moment $[M^{\zeta},M^{\xi},M^{\eta}]\cdot U''$
$\vec{M}_H$	external hydrodynamic moment per unit length
$\vec{M}_i$	moment per unit length due to internal fluid flow
M^n, M^b	bending moment projections along $\vec{n}$ and $\vec{b}$
$M^{\zeta}, M^{\xi}, M^{\eta}$	torsional and bending moments around $\vec{\xi}$ and $\vec{\eta}$

NOMENCLATURE (continued)

n	order of flexural mode
$\vec{n}$	unit vector in normal direction
p	internal overpressure due to well (i.e. total internal static pressure minus p_i'
p_i'	internal pressure due to gravity, $\rho_i g(h_i-y)$
p_o	external pressure due to gravity, $\rho_w g(h_w-y)$
P	tension in riser material
Q^ξ, Q^η	shear force in the $\vec{\xi}$ and $\vec{\eta}$ direction
r	radius
r_i, r_o	inner and outer radii
$\vec{R}$	position vector of an arbitrary point on the riser center-line
$\vec{R}_i$	position vector of the center of mass of a differential internal fluid element
Re_i	Reynold's number for internal fluid flow
s^*, s	stretched and unstretched length of the centerline
S	surface of the materials of the cross-section participating in bending
t	time
$\vec{t}$	unit vector in tangential direction
$T';T$	$T'=P+p_oA_o-p_i'A_i$; T = effective tension, see equation (II.91)
u	radial displacement in riser material
U	array of unit vectors $[\vec{i},\vec{j},\vec{k}]^T$
U''	array of unit vectors $[\vec{\zeta},\vec{\xi},\vec{\eta}]^T$

NOMENCLATURE (continued)

$\vec{v}$	absolute velocity of C, $\vec{v}=[v^{\zeta},v^{\xi},v^{\eta}]\cdot U''$
$\dot{V}_i$	constant volume flow rate of the internal fluid
W	effective weight per unit length, $W=W_R+W_i+W_b-B_b-B^*$
W_a	average effective weight per unit length in water
W_b,W_i,W_R	buoyancy module material, internal fluid and riser material weights per unit length
x,y,z	coordinates of C in the inertial frame
x^1,x^2,x^3	coordinates, see equation B.2
X_C,Y_C	coordinates of C
α^i	direction cosines of $\vec{t}$ with respect to U
β^i	direction cosines of $\vec{n}$ with respect to U
γ^i	direction cosines of $\vec{b}$ with respect to U
$\vec{\Delta}$	structural damping force per unit length
$\varepsilon_r,\varepsilon_\theta,\varepsilon_z$	strains in radial, circumferential and axial directions
$\vec{\Theta}$	structural damping moment per unit length
K	curvature of the centerline
ν	Poisson's ratio
ν_i	kinematic viscosity of the internal fluid
ρ_i,ρ_w	internal fluid and salt water densities
$\sigma_r,\sigma_\theta,\sigma_z$	stresses in radial, circumferential and axial directions
τ	geometric torsion
ϕ,θ,ψ	Euler angles, see Figure II.1
ϕ^f,θ^f,ψ^f	Euler angles of internal fluid element

NOMENCLATURE (continued)

Φ	orientation angle, see Figure A.1
$d\vec{\Phi}$	vector of the angle of infinitesimal rotation of the system $C\vec{\zeta}\vec{\xi}\vec{\eta}$
$\vec{\omega}, \vec{\omega}^f$	absolute angular velocity of $C\vec{\zeta}\vec{\xi}\vec{\eta}$ frame, $\vec{\omega}=[\omega^\zeta,\omega^\xi,\omega^\eta]\cdot U''$, and of internal fluid element $\vec{\omega}^f=[\omega^{f,\zeta},\omega^{f,\xi},\omega^{f,\eta}]\cdot U''$
$\vec{\Omega}$	vector rate of rotation of $C\vec{\zeta}\vec{\xi}\vec{\eta}$ frame along the rod, $\vec{\Omega}=[\Omega^\zeta,\Omega^\xi,\Omega^\eta]\cdot U''$

CHAPTER I

INTRODUCTION AND OUTLINE

Compliant risers are assemblages of pipes with very small overall bending rigidity used to convey oil from the ocean floor or a subsurface buoy to a surface platform, see Figures I.1 to I.4. A compliant riser is permitted to acquire large static deformations because of its small bending rigidity and readjusts its configuration in response to large motions of the supporting platforms, to which it is rigidly connected, without excessive stressing. Compliant risers have been used successfully in protected waters in buoy loading stations for tankers. Extension of shallow water concepts to deepwater have been proposed by the industry [1 to 8] as alternatives to conventional production risers because they simplify the overall production system.

The purpose of this work is to provide a general non-linear mathematical model describing the global behavior of a compliant riser idealized as a slender non-rotationally uniform rod with bending, extensional and torsional degrees of freedom in three dimensions and which includes the effects of external and internal pressure and speed of the internal fluid on the system.

This work is organized as follows: Chapter II includes the development of the mathematical model, including

- model assumptions
- equilibrium equations
- constitutive relations
- the relations between the rate of rotation of the body system along the length, the Cartesian coordinates, acceleration, velocity and angular velocity of the riser with the Euler angles

- geometric compatibility relations
- the relations between the rate of rotation of the body system along the length with the angular velocity
- the relations between the time rate of change of the angular momentum per unit length with the angular velocities and accelerations
- estimation of the force and moment per unit length due to the internal flow
- analysis of the equilibrium equations in the local principal directions
- reduction of the governing equations to a first order system of partial differential equations
- boundary and initial conditions
- specialization of the general governing equations for planar response without torsion

Appendix A provides the definitions of the structural rigidities of a cross-section. Appendix B provides derivation of the constitutive relations in the local tangential, normal and binormal system to the centerline. Appendix C provides derivation of the constitutive relation between effective tension and extensional strain of the centerline.

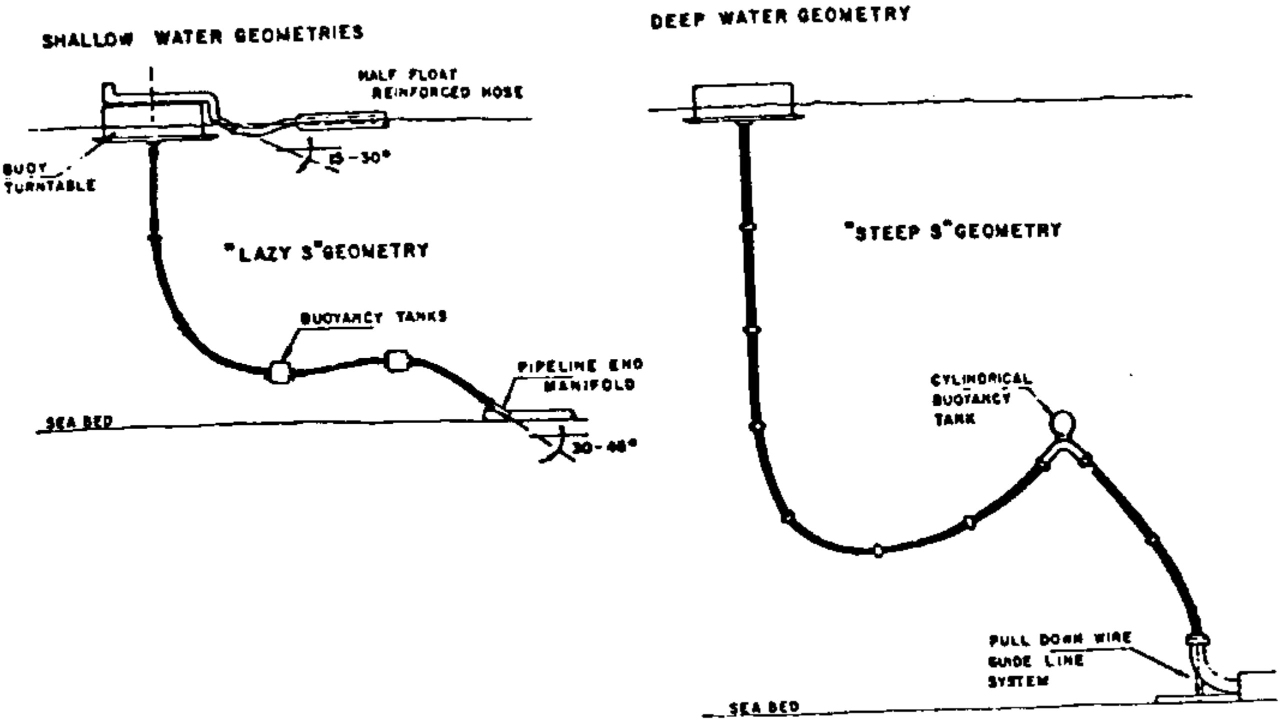

Figure I.1. Lazy and Steep S Geometries, adapted from [1]

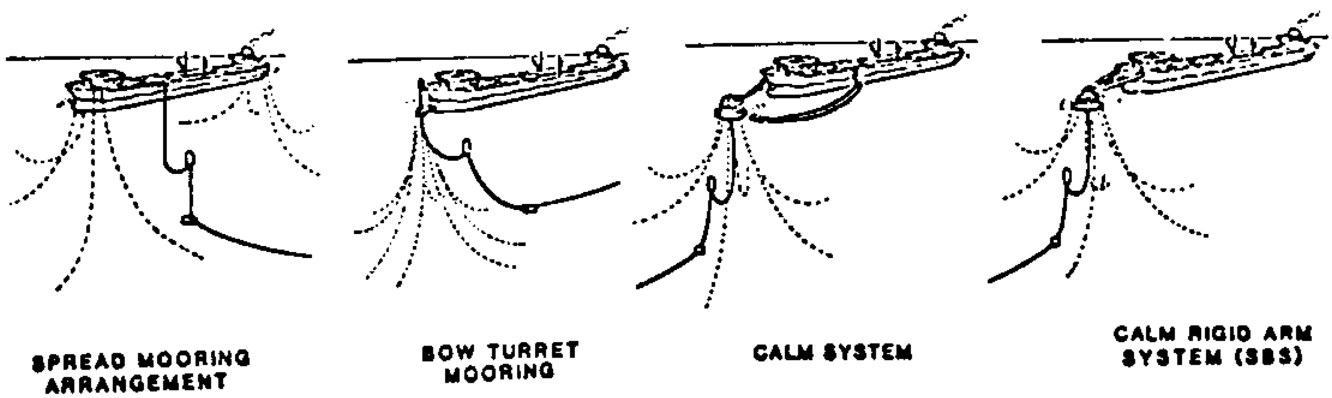

Figure I.2. Flexible Riser Production Concepts, adapted from [3]

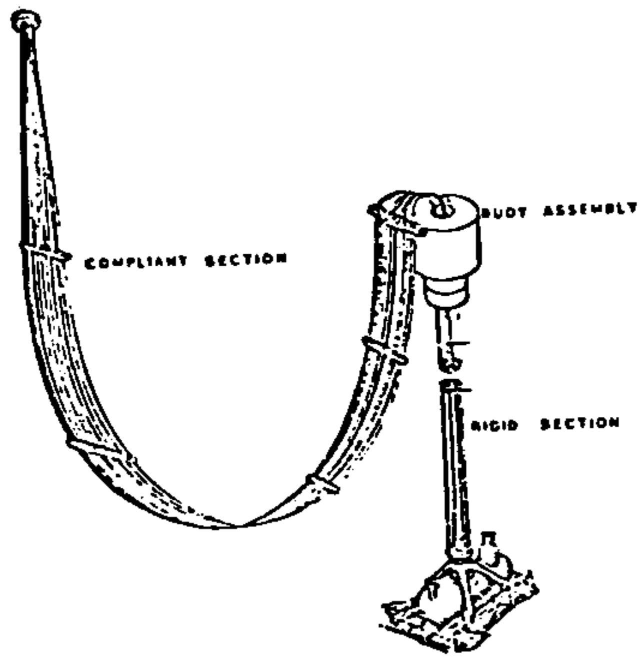

Figure I.3. Catenary Compliant Riser, adapted from [6]

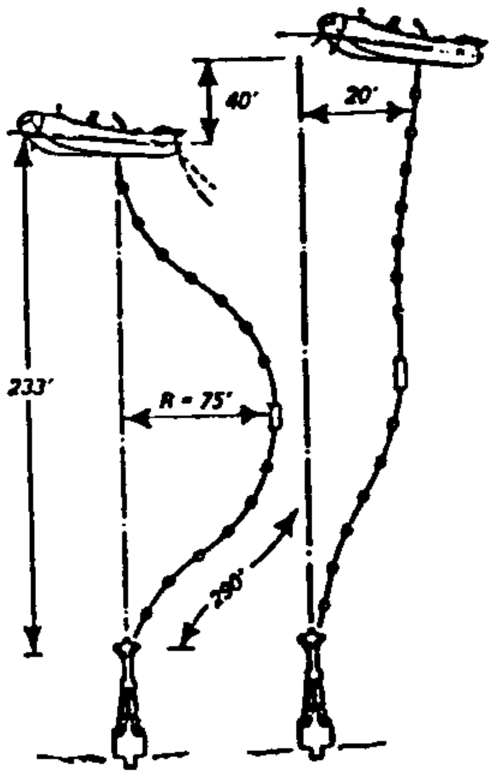

Figure I.4. Buoyant Compliant Riser, adapted from [7]

CHAPTER II

DEVELOPMENT OF THE MATHEMATICAL MODEL

II.1 MODEL ASSUMPTIONS

A mathematical model for the static behavior of slender elastic rods undergoing large deformations with small strains is given in Love [9] and Landau and Lifshitz [10]. The modification to account for dynamic effects and the presence of a heavy fluid inside and outside the tube modelled as a slender rod can be found in Nordgren [11] and Patrikalakis [12]. Methods for the computation of the motion of elastic rods with equal principal stiffnesses and with torque applied at the ends can be found in Nordgren [13,11] and without torque in Garrett [14].

In this work we extend the mathematical model derived in Nordgren [11] and Patrikalakis [12] to allow the computation of the motion of an assemblage of tubes modelled as a non-rotationally uniform slender elastic rod with space varying torque. The model derived here also accounts for the effects of steady internal flow in the non-linear regime. A related model allowing study of the effects of steady internal flow on the linear dynamics of planar naturally curved tubes can be found in Hill and Davis [15].

The basic assumptions of our model are listed below:

1. The compliant riser is modelled as a <u>single</u> non-rotationally uniform rod rather than as an assemblage of interacting rods or shells. We make this idealization in order to reduce the degrees of freedom and to allow analysis of the <u>global behavior</u> of our system with the currently available information on the structural characteristics

of such structures. It is noted that for some compliant riser configurations, such as the one proposed by Panicker and Yancey [6], the equations of the individual members composing the riser and the interactions between members need to be analyzed. Certain phenomena, for example, such as whirling instabilities of linear riser arrays, Blevins [16] and Ottesen Hansen and Panicker [17], necessitate this level of more detailed analysis.

2. The materials employed in the construction of different layers of compliant risers are assumed to be homogeneous, isotropic and linearly elastic.
3. Strains are assumed to remain uniformly small although deformations may become large.
4. Shearing deformations are neglected [9 to 15]. This is justified because they are of order $(Dn/L)^2$ compared to rotations of riser cross sections after bending, where D,L are the diameter and the length of the riser and n the order of the excited flexural mode. For typical configurations $D/L \ll 1$ and n is small; i.e., low frequencies are excited. This assumption implies that plane cross sections remain plane after bending and normal to the neutral axis as in the Rayleigh slender beam theory, see Crandall, et al. [18].
5. Thermal effects are neglected.

Assumption 1 implies strain continuity across layers of different materials in a given assemblage of tubes.

This idealization together with assumptions 2 to 5 allows the computation of equivalent bending, extensional and torsional rigidities of a particular cross section of the compliant riser, as it is shown in Appendix A. Two values of the bending rigidity, $EI^{\xi\xi}$ and $EI^{\eta\eta}$ are required for each cross section, where ξ and η are the centroidal principal axes of the cross section around which the bending rigidity is maximum and minimum, respectively. The term centroid, C, of a cross section denotes the moment centroid of the cross section with weighing factor the Young's modulus of the materials participating in bending, see Crandall, et al. [19]. In this work we also assume that

6. The centroid, C, defined above is also the mass centroid of the cross section.
7. The axes ζ, ξ and η are principal axes of the mass inertia of the cross section, where ζ is orthogonal to ξ and η at the centroid C.

Further theoretical and experimental research might be necessary to quantify the errors implied by the above list of assumptions, particularly assumptions 1 to 5.

II.2 EQUILIBRIUM EQUATIONS

We define a basic right-handed orthogonal Cartesian inertial reference frame Oxyz with corresponding unit vectors $\vec{i}$,$\vec{j}$ and $\vec{k}$, such that $\vec{j}$ is directed vertically upwards, and an orthogonal right-handed body system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ at each cross section of the rod. Point O can be chosen as any point fixed with respect to the earth. For convenience, however, we choose O to coincide with the centroid, C_o, of the lowest end cross section of the rod if C_o is fixed with respect to the earth or with its mean position if it is allowed to move. The unit vector $\vec{\zeta}$ is tangential to the centerline of the riser. The centerline is defined as the continuous line that joins all points C in the different cross sections of the rod. Each cross section can be identified by the unstretched arc length of the centerline measured from C_0. The vector $\vec{\zeta}$ points in the direction of increasing arc lengths. The directions of $\vec{\xi}$ and $\vec{\eta}$ are chosen in the previous section and point in such a way so that $\vec{\zeta}\vec{\xi}\vec{\eta}$ are a right-handed system. This system of axes is called the principal torsion-flexure system of axes of the rod at each point C of the centerline, Love [9].

The equations of equilibrium of forces and moments acting on a differential element ds of a compliant riser with centroid C are:

$$\vec{F}'_s - W\vec{j} + \vec{F}_H + \vec{F}_i + \vec{\Delta} = m\,\overrightarrow{{}^{\zeta}a} \qquad \text{(II.1)}$$

$$\vec{M}_s + \vec{\zeta} \times \vec{F}' + \vec{M}_H + \vec{M}_i + \vec{\Theta} = d\vec{H}_{R,C}/dt \qquad \text{(II.2)}$$

where $$\vec{F}' = [T', Q^{\xi}, Q^{\eta}].U'' \qquad \text{(II.3)}$$

$$U'' = [\vec{\zeta}, \vec{\xi}, \vec{\eta}]^T \qquad \text{(II.4)}$$

$$T' = P + p_o A_o - p_i' A_i \qquad \text{(II.5)}$$

$$p_o = \rho_w g(h_w - y) \tag{II.6}$$

$$p_i' = \rho_i g(h_i - y) \tag{II.7}$$

$$W = W_R + W_i + W_b - B_b - B^* \tag{II.8}$$

$$m^\zeta = (W_R + W_b)/g \tag{II.9}$$

$\vec{F}_H$ is the external hydrodynamic force per unit length excluding the effects of static pressure due to gravity.

$\vec{F}_i$ is the internal hydrodynamic force per unit length excluding the effects of static pressure due to gravity.

$\vec{M}$ restoring moment, $M^\zeta\vec{\zeta}+M^\xi\vec{\xi}+M^\eta\vec{\eta}$

$\vec{M}_H$ is the external moment per unit length

$\vec{M}_i$ is the moment per unit length due to the internal fluid flow

$\vec{\Lambda}$ and $\vec{\Theta}$ are structural damping force and moment per unit length

$\vec{a}$ absolute acceleration of C

$\vec{H}_{R,C}$ is the angular momentum per unit length of the riser material and buoyancy modules with respect to C

P	tension in the riser material
Q^{ξ},Q^{η}	shear forces in the $\vec{\xi}$ and $\vec{\eta}$ direction
p_o,p_i'	static water and internal fluid pressures due to gravity at elevation $y(s)$
ρ_w,ρ_i	salt water and inner fluid density
g	acceleration of gravity
h_w,h_i	salt water and internal fluid heights above C_o
W_R,W_i,W_b	Riser material, internal fluid and buoyancy module weights per unit length
B_b	Buoyancy per unit length due to buoyancy modules
B^*	Weight per unit length of displaced salt water by riser tubes
A_o,A_i	Total outer and inner cross sectional areas of the riser. A_i is assumed to be constant with s.

Subscript s denotes partial derivative with respect to s, the unstretched arc length of the centerline. d/dt denotes partial derivative with respect to time for a (vector) quantity expressed in the inertial frame. Equations (1) and (2) are valid within small strain theory $e=s_s^*-1$, where $s^*(s)$ is the stretched arc length of the centerline, see assumption 3 and Love [9].

II.3 ANALYSIS OF DEFORMATION-CONSTITUTIVE RELATIONS

It is convenient to analyze the governing equations (1) and (2) in the centroidal principal axes $C\vec{\zeta}$, $C\vec{\xi}$ and $C\vec{\eta}$, because the compliant riser is not rotationally uniform and because bending and torsion effects are included in our mathematical model. It is noted that the equations of conventional marine risers, Patrikalakis [12], Bernitsas [20,21] and cables, Triantafyllou [22], Triantafyllou and Bliek [23], Bliek [24] and Triantafyllou, et al. [25] have been analyzed in the local tangential, $\vec{t}=\vec{\zeta}$, normal, $\vec{n}$, and binormal, $\vec{b}$, directions to the centerline. The system $\vec{t},\vec{n},\vec{b}$ is convenient in the case of conventional risers, because the bending moment is directed exclusively in the binormal direction due to the rotational uniformity of the cross section, Love [9]. In the case of cables where bending effects are usually neglected and structural torsion is uncoupled from the other modes of deformation, it is sufficient to examine the deformations of the centerline, which can in turn be conveniently expressed in terms of the $\vec{t},\vec{n},\vec{b}$ system. The interesting relation between an analysis in the $C\vec{\zeta}\vec{\xi}\vec{\eta}$ and $C\vec{t}\vec{n}\vec{b}$ systems for a non-rotationally uniform slender rod is presented in Appendix B.

To describe flexural, extensional and torsional deformations of the rod, it is convenient to divide the rod into infinitesimal elements of stretched arc length ds*, each of which is bounded by two adjacent cross sections. To each cross section we attach a local body system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ defined in the previous section. It is assumed that if the centerline of the rod is rectilinear and no torsion is applied, all systems, $C\vec{\zeta}\vec{\xi}\vec{\eta}$, are mutually parallel for all C along the rod. At any fixed time t, any two adjacent systems $C\vec{\zeta}\vec{\xi}\vec{\eta}$ are rotated through an infinitesimal relative angle. It is known that an infinitesimal angle of rotation can be regarded as a vector parallel to the axis of rotation, Crandall, et al. [18]. Let $d\vec{\Phi}$

be the vector of the angle of infinitesimal rotation of a system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ at s*+ds* relative to the system at s* at a fixed time t. The components of $d\vec{\Phi}$ are the angles of rotation about each of the coordinate axes $\vec{\zeta},\vec{\xi},\vec{\eta}$. To describe the deformation we need to define the vector rate of rotation of the coordinate axes system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ along the rod, Love [9], Landau and Lifshitz [10]:

$$\vec{\Omega} = \vec{\Phi}_{s^*}$$

where subscript s* denotes partial derivative with respect to the stretched arc length s*. In all subsequent analysis, differentiations with respect to s*, to determine components of $\vec{\Omega}$, will be replaced by differentiation with respect to the unstretched arc length s of the centerline, because the extensional strain of the centerline is assumed small, e<<1, see assumption 3. A discussion of this approximation can be found in Love [9]. Therefore, consistent with equations (1) and (2) and our subsequent analysis, we will use

$$\vec{\Omega} = \vec{\Phi}_s \qquad \text{(II.10)}$$

For the choice of axes $\vec{\zeta},\vec{\xi}$ and $\vec{\eta}$ adopted in the previous section, the following constitutive relations between the restoring moment $\vec{M}$ and $\vec{\Omega}$ are valid as a result of the basic assumptions 1 to 5:

$$M^{\zeta} = GI^{P}\Omega^{\zeta}, M^{\xi} = EI^{\xi\xi}\Omega^{\xi},\ M^{\eta} = EI^{\eta\eta}\Omega^{\eta} \qquad \text{(II.11)}$$

where GI^{P}, $EI^{\xi\xi}$ and $EI^{\eta\eta}$ are the torsional and principal bending rigidities of the cross section. Estimates of these rigidities can be obtained with the method outlined in Appendix A. The constitutive relation for M^{ξ} and M^{η} are based on the basic approximation of slender rod theory, Love [9], according to which the extensional strain due to bending parallel $\vec{\zeta}$ at an arbitrary material point of the cross section with coordinates ξ and η, is given by $\Omega^{\xi}\eta-\Omega^{\eta}\xi$. Equations (A.5) and the above expression for the extensional strain due to Ω^{ξ} and Ω^{η} imply that this extensional strain does not produce a net force along $\vec{\zeta}$. An extensive discussion of the validity of (II.11) can be found in Love [9], pp. 389-395.

It is appropriate at this point to summarize the results of Appendix B concerning the analysis of bending moment, $M^{\xi}\vec{\xi}+M^{\eta}\vec{\eta}$ on the local normal and binormal vectors to the centerline for a non-rotationally uniform rod. The bending moment projections along $\vec{b}$ and $\vec{n}$ are given by

$$M^{b} = EI^{bb}\ K,\ M^{n} = -EI^{nb}\ K \tag{II.12}$$

where EI^{bb} is the bending rigidity of the cross section about $\vec{b}$, EI^{nb} is the cross product of bending rigidity about $\vec{n}$ and $\vec{b}$ and K the curvature of the centerline, Eisenhart [26]. Equation (II.12) implies, as stated earlier, that in rotationally uniform rods, where $EI^{nb}=0$, the bending moment is exclusively directed in the $\vec{b}$ direction; i.e. $M^{n}=0$. This fact is, for example, used in the derivations of Nordgren [13], Garrett [14], Patrikalakis [12] and Kim [27].

In order to complete the governing equations, we need to derive the constitutive relation between effective tension T and extensional strain, e, of the centerline. This is done in Appendix C, where it is shown that under a number of realistic assumptions

$$T = EA\ e \tag{II.13}$$

Relation (13) is an extension of the constitutive relation used in cable models, Goodman and Breslin [28], Triantafyllou [22], Bliek [24] and Triantafyllou, et al. [25] to include the effects of the internal pressure. The term effective tension is introduced in Section II.12 and Appendix C.

II.4 RELATIONS BETWEEN $\vec{\Omega}$ AND EULER ANGLES

In order to develop the above relation, we first introduce a set of Euler angles which define the orientation of the $C\vec{\zeta}\vec{\xi}\vec{\eta}$, system with respect to the inertial system $O\vec{i}\vec{j}\vec{k}$. Figure II.1 provides an illustration of the Euler angles ϕ,θ and ψ used in this work.

The first rotation ϕ is performed around $\vec{k}$ and brings $(\vec{\zeta}_1,\vec{\xi}_1,\vec{\eta}_1) = (\vec{i},\vec{j},\vec{k})$ to $(\vec{\zeta}_2,\vec{\xi}_2,\vec{\eta}_2)$ where $\vec{\eta}_2=\vec{\eta}_1=\vec{k}$. The second rotation θ is performed around $\vec{\xi}_2$ and brings $(\vec{\zeta}_2,\vec{\xi}_2,\vec{\eta}_2)$ to $(\vec{\zeta},\vec{\xi}_3,\vec{\eta}_3)$ where $\vec{\xi}_3=\vec{\xi}_2$. Finally, the third rotation ψ is performed around $\vec{\zeta}$ and brings $(\vec{\zeta},\vec{\xi}_3,\vec{\eta}_3)$ to $(\vec{\zeta},\vec{\xi},\vec{\eta})$. From these definitions we can see that our body axes system is singular when $\theta = \pm \pi/2$, see Crandall, et al. [18]. This however is not a problem because in our axes systems the value of θ is always small and near zero throughout the riser. This comes about from the fact that we expect the compliant riser to be deployed so that θ is approximately equal to zero at the lower end in order to avoid excessive stressing. The values of θ remain small throughout the riser length because in our inertial axes system definition we selected the predominant current direction to be in the $O\vec{i}\vec{j}$ plane.

The system of Euler angles defined above also has the advantage that one finite rotation ϕ about $\vec{k}$ provides a complete description of a general two dimensional problem in the $O\vec{i}\vec{j}$ plane in the absence of torque. In

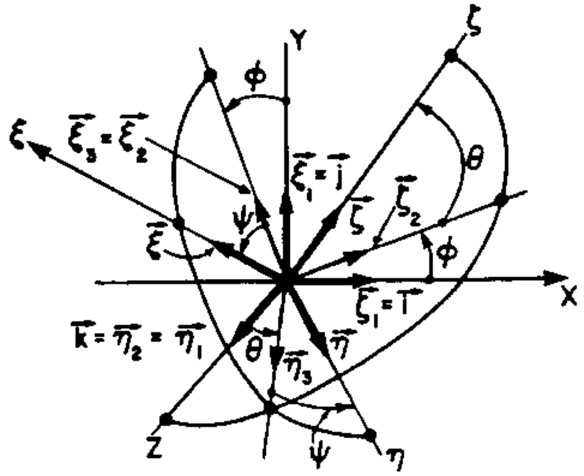

Figure II.1. Euler angles definition

addition, this system has the advantage that slightly non-planar configurations with torsion are described with one finite rotation ϕ and small rotations θ and ψ. Such configurations are very often encountered in compliant riser systems proposed to date. Finally, the introduction of Euler angles to describe the orientation of the body system facilitates the expression of the external hydrodynamic forces and moments on a non-rotationally uniform structure.

Following Goldstein [29], the complete 3x3 transformation matrix $C=[c_{ij}]$ between $U=[\vec{i},\vec{j},\vec{k}]^T$ and $U''=[\vec{\zeta},\vec{\xi},\vec{\eta}]^T$ defined by

$$U'' = C \cdot U \tag{II.14}$$

can be easily obtained as the triple product of the three separate rotations, each of which has a relatively simple matrix form. This procedure leads to

$$c_{11} = \cos\theta \ \cos\phi \tag{II.15.1}$$

$$c_{12} = \cos\theta \ \sin\phi \tag{II.15.2}$$

$$c_{13} = -\sin\theta \tag{II.15.3}$$

$$c_{21} = \sin\theta \ \sin\psi \ \cos\phi - \cos\psi \ \sin\phi \tag{II.15.4}$$

$$c_{22} = \sin\theta \ \sin\psi \ \sin\phi + \cos\psi \ \cos\phi \tag{II.15.5}$$

$$c_{23} = \cos\theta \, \sin\psi \tag{II.15.6}$$

$$c_{31} = \sin\theta \, \cos\psi \, \cos\phi + \sin\psi \, \sin\phi \tag{II.15.7}$$

$$c_{32} = \sin\theta \, \cos\psi \, \sin\phi - \sin\psi \, \cos\phi \tag{II.15.8}$$

$$c_{33} = \cos\theta \, \cos\psi \tag{II.15.9}$$

We are now in a position to relate the components of $\vec{\Omega}$ with ϕ_s, θ_s, ψ_s and the Euler angles ϕ, θ and ψ. These relations can be obtained by noting that

$$\vec{\Omega} = \vec{\phi}_s \vec{k} + \vec{\theta}_s \vec{\xi}_2 + \psi_s \vec{\zeta} \tag{II.16}$$

$$\vec{\xi}_2 = [-\sin\phi, \cos\phi, 0] \cdot U \tag{II.17}$$

where $\vec{\xi}_2$ is defined in Figure II.1, and using equation (II.10) and (II.14).

The following results are derived by projecting (II.16) on $\vec{\zeta}, \vec{\xi}$ and $\vec{\eta}$ respectively:

$$\Omega^{\zeta} = \psi_s - \phi_s \sin\theta \tag{II.18}$$

$$\Omega^{\xi} = \theta_s \cos\psi + \phi_s \cos\theta \sin\psi \tag{II.19}$$

$$\Omega^{\eta} = -\theta_s \sin\psi + \phi_s \cos\theta \cos\psi \tag{II.20}$$

II.5 RELATIONS BETWEEN THE CARTESIAN COORDINATES OF C AND THE EULER ANGLES

Let

$$\vec{R} = [x,y,z] \cdot U \tag{II.21}$$

be the position vector of an arbitrary point C on the riser centerline. The tangential vector, $\vec{\zeta}$, to the centerline can be found by

$$\vec{\zeta} = \vec{R}_{s*} \tag{II.22}$$

where s*(s,t) is the stretched arc length corresponding to point C with Lagrangian coordinate, s, Eisenhart [26]. Using

$$e = s^*_s - 1 \tag{II.23}$$

we find that

$$\vec{\zeta} = \vec{R}_s/(1+e) \tag{II.24}$$

which by using (II.15.1) - (II.15.3) leads to

$$x_s = (1+e) \cos\theta \cos\phi \tag{II.25}$$

$$y_s = (1+e) \cos\theta \sin\phi \tag{II.26}$$

$$z_s = -(1+e) \sin\theta \tag{II.27}$$

II.6 RELATIONS BETWEEN ACCELERATION, VELOCITY AND ANGULAR VELOCITY AND EULER ANGLES

The absolute acceleration $\vec{a}$ of point C can, of course, be evaluated as

$$\vec{a} = [x_{tt}, y_{tt}, z_{tt}]\cdot U \tag{II.28}$$

Alternatively, we may calculate $\vec{a}$ by using the components of the absolute velocity, $\vec{v}$, of C and angular velocity, $\vec{\omega}$, of the $C\vec{\zeta}\vec{\xi}\vec{\eta}$ system in the local $\vec{\zeta},\vec{\xi},\vec{\eta}$ directions. This formulation allows an easier evaluation of the angular momentum (assumption 7) and of the external force and moment terms for a non-rotationally uniform rod. The simplification of the external force and moment terms is based on the further assumption that the $\vec{\zeta},\vec{\xi},\vec{\eta}$ directions are also principal directions for the added mass/inertia tensor, see Newman [30].

We, therefore, let the absolute velocity of C be analyzed as

$$\vec{v} = [v^{\zeta}, v^{\xi}, v^{\eta}]\cdot U'' \tag{II.29}$$

and, therefore, following Crandall, et al. [18], we obtain

$$\vec{a} = [v^{\zeta}_t, v^{\xi}_t, v^{\eta}_t]\cdot U'' + \vec{\omega}\mathrm{x}[v^{\zeta}, v^{\xi}, v^{\eta}]\cdot U'' \tag{II.30}$$

where $\vec{\omega}$ is the absolute angular velocity of the $\vec{\zeta}\vec{\xi}\vec{\eta}$ system. By analyzing $\vec{\omega}$ and $\vec{a}$ as

$$\vec{\omega} = [\omega^{\zeta},\omega^{\xi},\omega^{\eta}]\cdot U'' \qquad \text{(II.31)}$$

$$\vec{a} = [a^{\zeta}, a^{\xi}, a^{\eta}]\cdot U'' \qquad \text{(II.32)}$$

we obtain:

$$a^{\zeta} = v_t^{\zeta} + \omega^{\xi}v^{\eta} - \omega^{\eta}v^{\xi} \qquad \text{(II.33)}$$

$$a^{\xi} = v_t^{\xi} + \omega^{\eta}v^{\zeta} - \omega^{\zeta}v^{\eta} \qquad \text{(II.34)}$$

$$a^{\eta} = v_t^{\eta} + \omega^{\zeta}v^{\xi} - \omega^{\xi}v^{\zeta} \qquad \text{(II.35)}$$

The components of the angular velocity $\vec{\omega}$ in the $\vec{\zeta},\vec{\xi}$ and $\vec{\eta}$ directions can be easily found in terms of time derivatives of Euler angles and the Euler angles themselves. Following Landau and Lifshitz [31], we note that

$$\vec{\omega} = \phi_t\vec{k} + \theta_t\vec{\xi}_2 + \psi_t\vec{\zeta} \qquad \text{(II.36)}$$

Projecting (II.36) on the local $\vec{\zeta},\vec{\xi},\vec{\eta}$ directions and using (II.17) and (II.14), we obtain:

$$\omega^{\zeta} = \psi_t - \phi_t \sin\theta \tag{II.37}$$

$$\omega^{\xi} = \theta_t \cos\psi + \phi_t \cos\theta \sin\psi \tag{II.38}$$

$$\omega^{\eta} = -\theta_t \sin\psi + \phi_t \cos\theta \cos\psi \tag{II.39}$$

Note the similarity of (II.37) to (II.39) to (II.18) to (II.20), respectively.

II.7 GEOMETRIC COMPATIBILITY RELATIONS

These are relations connecting the space derivatives of the $\vec{\zeta},\vec{\xi},\vec{\eta}$ components of $\vec{v}$, the components of $\vec{\Omega},\vec{\omega}$ and the extensional strain, e, of the centerline. These relations are easily obtained by rewriting (II.24) as $\vec{R}_s=(1+e)\vec{\zeta}$, taking a time partial derivative, substituting $\vec{R}_t$ by $\vec{v}$ and using the fact that $\vec{\zeta}_t=\vec{\omega}\times\vec{\zeta}$, see Crandall, et al. [18]. This procedure leads to

$$\vec{v}_s = e_t\vec{\zeta} + (1+e)\,(\vec{\omega}\times\vec{\zeta}) \tag{II.40}$$

However, using (II.29) and the relations

$$\vec{\zeta}_s = \vec{\Omega}\times\vec{\zeta},\ \vec{\xi}_s = \vec{\Omega}\times\vec{\xi},\ \vec{\eta}_s = \vec{\Omega}\times\vec{\eta} \tag{II.41}$$

we can also write that

$$\vec{v}_s = [v^{\zeta}_s, v^{\xi}_s, v^{\eta}_s]\cdot U'' + \vec{\Omega}\times[v^{\zeta}, v^{\xi}, v^{\eta}]\cdot U'' \tag{II.42}$$

Relations (II.41) can be found in component form in Love [9]. They can be easily proved by noting that for a fixed time t, the changes of vectors $\vec{\zeta}, \vec{\xi}$ and $\vec{\eta}$ between two adjacent cross sections of the centerline can be written as

$$d\vec{\zeta} = d\vec{\Phi}\times\vec{\zeta},\ d\vec{\xi} = d\vec{\Phi}\times\vec{\xi},\ d\vec{\eta} = d\vec{\Phi}\times\vec{\eta} \tag{II.43}$$

where $d\vec{\Phi}$ is the vector of angle of infinitesimal rotation of the $\vec{\zeta}\vec{\xi}\vec{\eta}$ system at s+ds relative to the system $\vec{\zeta}\vec{\xi}\vec{\eta}$ at s, see Landau and Lifshitz [10]. Dividing (II.43) by ds and using (II.10), relations (II.41) are obtained. Introducing

$$\vec{\Omega} = [\Omega^{\zeta}, \Omega^{\xi}, \Omega^{\eta}]\cdot U'' \tag{II.44}$$

in (II.42) and eliminating $\vec{v}_s$ between (II.40) and (II.42), we obtain the compatibility relations in component form in the $\vec{\zeta}, \vec{\xi}$ and $\vec{\eta}$ directions.

$$v_s^{\zeta} + \Omega^{\xi}v^{\eta} - \Omega^{\eta}v^{\xi} = e_t \tag{II.45}$$

$$v_s^{\xi} + \Omega^{\eta}v^{\zeta} - \Omega^{\zeta}v^{\eta} = (1+e)\omega^{\eta} \tag{II.46}$$

$$v_s^{\eta} + \Omega^{\zeta}v^{\xi} - \Omega^{\xi}v^{\zeta} = -(1+e)\omega^{\xi} \tag{II.47}$$

Relations similar to (II.45) to (II.47) in the local $\vec{t}$,$\vec{n}$ and $\vec{b}$ directions of the centerline of a cable can be found in Bliek [24] and Triantafyllou, et al. [25].

II.8 RELATION BETWEEN $\vec{\Omega}$ AND $\vec{\omega}$

First we combine relations (II.18) to (II.20) to obtain:

$$\phi_s = (\Omega^{\xi}\sin\psi + \Omega^{\eta}\cos\psi)/\cos\theta \tag{II.48}$$

$$\theta_s = \Omega^{\xi}\cos\psi - \Omega^{\eta}\sin\psi \tag{II.49}$$

$$\psi_s = \Omega^{\zeta} + \phi_s\sin\theta \tag{II.50a}$$

Due to (II.48) relation (II.50a) can be also rewritten as

$$\psi_s = \Omega^\zeta + \tan\theta(\Omega^\xi \sin\psi + \Omega^\eta \cos\psi) \tag{II.50b}$$

In a similar manner relations (II.37) to (II.39) can be combined to give:

$$\phi_t = (\omega^\xi \sin\psi + \omega^\eta \cos\psi)/\cos\theta \tag{II.51}$$

$$\theta_t = \omega^\xi \cos\psi - \omega^\eta \sin\psi \tag{II.52}$$

$$\psi_t = \omega^\zeta + \phi_t \sin\theta \tag{II.53a}$$

Due to (II.51), relation (II.53a) can be also rewritten as

$$\psi_t = \omega^\zeta + \tan\theta(\omega^\xi \sin\psi + \omega^\eta \cos\psi) \tag{II.53b}$$

Relations (II.48), (II.50b), (II.51) and (II.53b) provide an explicit indication that $\theta = \pm\ \pi/2$ is a singular point of the ϕ,θ,ψ set of Euler axes used in this work as stated in Section II.4. However, for the reasons given in the above Section, θ will be substantially different from $\pm\ \pi/2$ for all configurations studied here.

The relations between $\vec{\Omega}$ and $\vec{\omega}$ can be obtained by taking the partial derivative with respect to time of equations (II.18) to (II.20) and using (II.48) to (II.53) to eliminate the partial derivatives of the Euler angles with respect to s and t. This procedure leads to:

$$\Omega_t^{\zeta} = \omega_s^{\zeta} + \Omega^{\xi}\omega^{\eta} - \Omega^{\eta}\omega^{\xi} \tag{II.54}$$

$$\Omega_t^{\xi} = \omega_s^{\xi} + \Omega^{\eta}\omega^{\zeta} - \Omega^{\zeta}\omega^{\eta} \tag{II.55}$$

$$\Omega_t^{\eta} = \omega_s^{\eta} + \Omega^{\zeta}\omega^{\xi} - \Omega^{\xi}\omega^{\zeta} \tag{II.56}$$

II.9 EVALUATION OF $d\vec{H}_{R,C}/dt$

We analyze the vector, $\vec{H}_{R,C}$, the angular momentum of the riser material per unit length with respect to the center of mass and centroid of each differential riser element in the local $\vec{\zeta}$,$\vec{\xi}$ and $\vec{\eta}$ directions which, due to assumption 7, are also principal axes of the mass inertia of the element:

$$\vec{H}_{R,C} = [H_{R,C}^{\zeta}, H_{R,C}^{\xi}, H_{R,C}^{\eta}]\cdot U'' \tag{II.57}$$

where

$$H^{\zeta}_{R,C} = J_R{}^{\zeta\zeta}\omega^{\zeta} \tag{II.58}$$

$$H^{\xi}_{R,C} = J_R{}^{\xi\xi}\omega^{\xi} \tag{II.59}$$

$$H^{\eta}_{R,C} = J_R{}^{\eta\eta}\omega^{\eta} \tag{II.60}$$

see Crandall, et al. [18]. As with equation (II.30), we obtain:

$$d\vec{H}_{R,C}/dt = [H^{\zeta}_{R,C_t}, H^{\xi}_{R,C_t}, H^{\eta}_{R,C_t}]\cdot U'' + \vec{\omega}x[H^{\zeta}_{R,C}, H^{\xi}_{R,C}, H^{\eta}_{R,C}]\cdot U'' \tag{II.61}$$

and therefore analyzing in the $\vec{\zeta}$,$\vec{\xi}$ and $\vec{\eta}$ directions we obtain:

$$dH^{\zeta}_{R,C}/dt = J^{\zeta\zeta}_{R}\omega^{\zeta}_{t} + (J^{\eta\eta}_{R}-J^{\xi\xi}_{R})\omega^{\xi}\omega^{\eta} \tag{II.62}$$

$$dH^{\xi}_{R,C}/dt = J^{\xi\xi}_{R}\omega^{\xi}_{t} + (J^{\zeta\zeta}_{R}-J^{\eta\eta}_{R})\omega^{\eta}\omega^{\zeta} \tag{II.63}$$

$$dH^{\eta}_{R,C}/dt = J^{\eta\eta}_{R}\omega^{\eta}_{t} + (J^{\xi\xi}_{R}-J^{\zeta\zeta}_{R})\omega^{\zeta}\omega^{\xi} \tag{II.64}$$

II.10 ESTIMATION OF THE FORCE $\vec{F}_i$ PER UNIT LENGTH DUE TO INTERNAL FLOW

Let $\vec{R}_i$ be the position vector of the center of mass of a differential internal fluid element defined as an infinitesimal cylinder of cross sectional area A_i, assumed constant with s, and height δs. We further assume that the line of the centers of mass of all such fluid elements coincides with the centerline of the riser and that W_i is constant with s. For each internal fluid element, we introduce a Lagrangian coordinate s_0 which is equal to the position $s=s_0$ of the fluid element at some fixed time t_0, see Crandall, et al. [18]. Fluid elements possess a mean flow velocity, c, where $c=\dot{V}_i/A_i$ and $\dot{V}_i$ is the constant volume flow rate of the internal fluid. At some time t, the position, s, of the fluid element is given by:

$$s = s_0 + c(t-t_0) \tag{II.65}$$

This relation is valid for small extensional strains of riser centerline, e<<1, and is, therefore, compatible with the degree of approximation implied by assumption 3.

From compatibility of the internal fluid flow and the riser motion, we obtain

$$\vec{R}_i(s_0,t) = \vec{R}(s,t) \tag{II.66}$$

where s is given by (II.65) and $\vec{R}(s,t)$ is the position vector of the riser centerline with respect to the inertial system. The compatibility relation (II.16) is also implied in the derivation of Hill and Davis [15], although they later restrict their attention to the linearized equations of motion.

Equilibrium of forces for an internal fluid element leads to:

$$\rho_i A_i \vec{R}_{i_{tt}}(s_o,t) = -\vec{F}_i(s,t) - A_i(p\vec{\zeta})_s \tag{II.67}$$

where subscript t denotes derivative with respect to time. The left-hand side of equation (67) is the time rate of change of momentum per unit length of the internal fluid element in the inertial system. The term $-\vec{F}_i(s,t)$ is force per unit length from the riser inner walls on the fluid element which includes normal pressure and frictional components. The term $-A_i(p\vec{\zeta})_s$ is the contribution of the overpressure, p, on the two end cross sections of the differential element. The contribution of gravity forces to the pressure and the overall equilibrium of forces is properly taken into account in equation (1) due to the definition of effective weight W and T'. Therefore, p, is indeed an overpressure resulting from the pressure of the well. The value of p varies little with s for the speeds and the geometries of interest in compliant risers. For a cylindrical tube

$$p_s \approx -\frac{1}{D_i}\frac{\rho_i}{2}c^2 \begin{cases} 64/Re_i & \text{if } Re_i \le 2300 \\ \lambda & \text{if } Re_i > 2300 \end{cases}$$

where $Re_i = cD_i/\nu_i$, and λ depends upon Re_i and roughness, Schlichting [32]. The value of λ is always approximately below 0.08. Typical values indicate that $p_s L/p \ll 1$. Equation (II.67) allows the computation of $\vec{F}_i$ in terms of the gross internal flow parameters ρ_i, A_i, c and p and $\vec{R}(s,t)$ if the compatibility relation (II.66) is employed to calculate $\vec{R}_{i_{tt}}(s_o,t)$. Following Crandall, et al. [18] we find from (II.66) that

$$\vec{R}_{i_t} = \vec{R}_t + c\vec{R}_s \tag{II.68}$$

$$\vec{R}_{i_{tt}} = \vec{R}_{tt} + 2c\vec{R}_{st} + c^2\vec{R}_{ss} \tag{II.69}$$

In addition we may use the relation

$$\vec{\zeta} \simeq \vec{R}_s \tag{II.70}$$

valid for small extensional strains, e<<1, see assumption 3;

$$\vec{\zeta}_t = \vec{\omega} x \vec{\zeta} \tag{II.71}$$

and
$$\vec{a} = \vec{R}_{tt} \tag{II.72}$$

where $\vec{\omega}$ is the absolute angular velocity of $C\vec{\zeta}\vec{\xi}\vec{\eta}$, to obtain:

$$\vec{R}_{i_{tt}} = \vec{a} + 2c\vec{\omega}x\vec{\zeta} + c^2\vec{\zeta}_s \tag{II.73}$$

Relations (II.73) and (II.67) imply that

$$\vec{F}_i(s,t) = -\rho_i A_i(\vec{a}+2c\vec{\omega}x\vec{\zeta}) - A_i[(p+\rho_i c^2)\vec{\zeta}]_s \tag{II.74}$$

The linearized versions of (II.73) and (II.74) can be found in Hill and Davis [15] for a planar naturally curved tube of constant initial curvature.

II.11 ESTIMATION OF THE MOMENT $\vec{M}_i$ PER UNIT LENGTH DUE TO INTERNAL FLOW

The derivations of this Section proceed as in Section II.10. We start by using the Lagrangian coordinate s_0 to identify internal fluid elements, so that the position of the fluid element at time t_0 is $s=s_0$. At time $t=t_0$, we define the Euler angles ϕ^f, θ^f and ψ^f of the fluid element to be equal to the Euler angles of the riser at $s=s_0$; i.e.

$$\phi^f(s_0,t_0)=\phi(s_0,t_0);\theta^f(s_0,t_0)=\theta(s_0,t_0),\psi^f(s_0,t_0)=\psi(s_0,t_0) \qquad \text{(II.75)}$$

At time t, we assume the following compatibility relations of the internal fluid flow and the riser motion:

$$\phi^f(s_0,t) = \phi(s,t) \qquad \text{(II.76)}$$

$$\theta^f(s_0,t) = \theta(s,t) \qquad \text{(II.77)}$$

$$\psi^f(s_0,t) = \psi(s,t) \qquad \text{(II.78)}$$

where s is given by (II.65). Relation (II.78) presupposes a non-circular fluid cross section or more than one circular cross section. For one circular tube, relation (II.78) is not valid in general but it could be adopted in order to decrease the degrees of freedom of the system.

Given that angular velocities are obtained from combinations of first time derivatives of the Euler angles, the angular velocity, $\vec{\omega}^f$, of a fluid element can be determined by

$$\vec{\omega}^f(s_0,t) = \vec{\omega}(s,t) + c\vec{\Omega}(s,t) \qquad \text{(II.79)}$$

where s is given by (II.65). In the derivation of (II.79), relations (II.18) to (II.20), (II.37) to (II.39) and (II.76) to (II.78) have been used. In component form equation (II.79) gives:

$$\omega^{f,\zeta} = \omega^{\zeta} + c\Omega^{\zeta} \tag{II.80.1}$$

$$\omega^{f,\xi} = \omega^{\xi} + c\Omega^{\xi} \tag{II.80.2}$$

$$\omega^{f,\eta} = \omega^{\eta} + c\Omega^{\eta} \tag{II.80.3}$$

where the dependencies on s_0, s and t in (II.79) have been omitted for brevity. Relations (II.80) allow the calculation of the $\vec{\zeta}$, $\vec{\xi}$ and $\vec{\eta}$ components of the angular fluid element acceleration by:

$$\omega^{f,\zeta}_{t} = \omega^{\zeta}_{t} + c\omega^{\zeta}_{s} + c\Omega^{\zeta}_{t} + c^{2}\Omega^{\zeta}_{s} \tag{II.81.1}$$

$$\omega^{f,\xi}_{t} = \omega^{\xi}_{t} + c\omega^{\xi}_{s} + c\Omega^{\xi}_{t} + c^{2}\Omega^{\xi}_{s} \tag{II.81.2}$$

$$\omega^{f,\eta}_{t} = \omega^{\eta}_{t} + c\omega^{\eta}_{s} + c\Omega^{\eta}_{t} + c^{2}\Omega^{\eta}_{s} \tag{II.81.3}$$

It is convenient to eliminate ω^{ζ}_{s}, ω^{ξ}_{s} and ω^{η}_{s} from (II.81) using (II.54) to (II.56) and, therefore, obtain:

$$\omega^{f,\zeta}_t = \omega^\zeta_t + 2c\Omega^\zeta_t + c(\Omega^\eta\omega^\xi - \Omega^\xi\omega^\eta) + c^2\Omega^\zeta_s \qquad \text{(II.82.1)}$$

$$\omega^{f,\xi}_t = \omega^\xi_t + 2c\Omega^\xi_t + c(\Omega^\zeta\omega^\eta - \Omega^\eta\omega^\zeta) + c^2\Omega^\xi_s \qquad \text{(II.82.2)}$$

$$\omega^{f,\eta}_t = \omega^\eta_t + 2c\Omega^\eta_t + c(\Omega^\xi\omega^\zeta - \Omega^\zeta\omega^\xi) + c^2\Omega^\eta_s \qquad \text{(II.82.3)}$$

Relations (II.80) and (II.82) allow calculation of the angular momentum per unit length, $\vec{H}_{i,C}$ of a differential fluid element in the riser with respect to C. In addition, since the moment per unit length of the normal pressure forces on the two cross sections of the fluid element, $-\vec{\zeta} \times pA_i\vec{\zeta}$, is zero, equilibrium of moments for the fluid element gives

$$d\vec{H}_{i,c}/dt = -\vec{M}_i \qquad \text{(II.83)}$$

where $\vec{M}_i$ is the moment exerted by the internal flow on the riser tubes, appearing in equation (II.2). Assuming that $\vec{\zeta}$, $\vec{\xi}$ and $\vec{\eta}$ are the principal axes of mass inertia of the fluid element, we may write that the components of

$$\vec{H}_{i,c} = [H_{i,c}^{\zeta},\ H_{i,c}^{\xi},\ H_{i,c}^{\eta}]\cdot U'' \qquad \text{(II.84)}$$

are given by

$$H_{i,c}^{\zeta} = J_i^{\zeta\zeta}\omega^{f,\zeta} \tag{II.85}$$

$$H_{i,c}^{\xi} = J_i^{\xi\xi}\omega^{f,\xi} \tag{II.86}$$

$$H_{i,c}^{\eta} = J_i^{\eta\eta}\omega^{f,\eta} \tag{II.87}$$

As with equations (II.62) to (II.64) we obtain

$$dH_{i,c}^{\zeta}/dt = J_i^{\zeta\zeta}\,\omega_t^{f,\zeta} + (J_i^{\eta\eta}-J_i^{\xi\xi})\omega^{f,\xi}\omega^{f,\eta} \tag{II.88}$$

$$dH_{i,c}^{\xi}/dt = J_i^{\xi\xi}\omega_t^{f,\xi} + (J_i^{\zeta\zeta}-J_i^{\eta\eta})\omega^{f,\eta}\omega^{f,\zeta} \tag{II.89}$$

$$dH_{i,c}^{\eta}/dt = J_i^{\eta\eta}\omega_t^{f,\eta} + (J_i^{\xi\xi}-J_i^{\zeta\zeta})\,\omega^{f,\zeta}\omega^{f,\xi} \tag{II.90}$$

where relations (II.80) and (II.82) may be used to express the components of the time rate of change of the fluid angular momentum in terms of $\vec{\omega}$ and $\vec{\Omega}$.

II.12 REDUCED FORM OF THE GOVERNING EQUATIONS

Equations (II.74) and (II.83) can be used to eliminate $\vec{F}_i$ and $\vec{M}_i$ from equations (II.1) and (II.2). Inspection of equations (II.1), (II.3), (II.5) and (II.74) leads to the conclusion that the following definitions provide considerable simplification of equation (II.1):

$$T = T' - (p+\rho_i c^2)A_i \tag{II.91.1}$$

or using (II.5)

$$T = P+p_0A_0-(p_i'+p+\rho_i c^2)A_i \tag{II.91.2}$$

$$\vec{F} = [T,Q^\xi,Q^\eta]\cdot U'' \tag{II.92}$$

$$m = m^\zeta+\rho_i A_i \tag{II.93}$$

The simplified form of the equation of forces, (II.1), is:

$$\vec{F}_S-W\vec{j}+\vec{F}_H+\vec{\Delta} = m\vec{a}+2c\rho_i A_i\ \vec{\omega}\times\vec{\zeta} \tag{II.94}$$

In addition, defining

$$\vec{H}_c = \vec{H}_{R,c} + \vec{H}_{i,c} \tag{II.95}$$

and using (II.83) and the fact that $\vec{\zeta}\times\vec{F}' = \vec{\zeta}\times\vec{F}$, the equation of moments, (II.2), obtains the simplified form:

$$\vec{M}_S+\vec{\zeta}\times\vec{F} + \vec{M}_H+\vec{\Theta} = d\vec{H}_c/dt \tag{II.96}$$

The new term, T, appearing in equations (II.91) and (II.92) is called effective tension and is a generalization of the term used in Bernitsas [20,21,33], Nordgren [11] and Patrikalakis [12] to include the effects of the internal flow.

The components of $d\vec{H}_C/dt$ in the $\vec{\zeta}$, $\vec{\xi}$ and $\vec{\eta}$ directions can be obtained by combining equations (II.62) to (II.64), (II.88) to (II.90) and (II.95). Introducing the following definitions:

$$J^{\zeta\zeta} = J_R^{\zeta\zeta} + J_i^{\zeta\zeta} \tag{II.97}$$

$$J^{\xi\xi} = J_R^{\xi\xi} + J_i^{\xi\xi} \tag{II.98}$$

$$J^{\eta\eta} = J_R^{\eta\eta} + J_i^{\eta\eta} \tag{II.99}$$

and also using (II.80) and (II.82), we obtain

$$\begin{aligned} \frac{dH_c^\zeta}{dt} = {} & c^2[J_i^{\zeta\zeta}\Omega_s^\zeta + (J_i^{\eta\eta}-J_i^{\xi\xi})\Omega^\xi\Omega^\eta] + \\ & + J^{\zeta\zeta}\omega_t^\zeta + (J^{\eta\eta}-J^{\xi\xi})\omega^\xi\omega^\eta + \\ & c[J_i^{\zeta\zeta}(2\Omega_t^\zeta + \Omega^\eta\omega^\xi-\Omega^\xi\omega^\eta) + (J_i^{\eta\eta}-J_i^{\xi\xi})(\Omega^\eta\omega^\xi+\Omega^\xi\omega^\eta)] \end{aligned} \tag{II.100}$$

The corresponding expressions in the $\vec{\xi}$ and $\vec{\eta}$ directions can be obtained cyclically:

$$\frac{dH_c^{\xi}}{dt} = c^2[J_i^{\xi\xi}\Omega_s^{\xi} + (J_i^{\zeta\zeta}-J_i^{\eta\eta})\Omega^{\eta}\Omega^{\zeta}] +$$

$$+ J^{\xi\xi}\omega_t^{\xi} + (J^{\zeta\zeta}-J^{\eta\eta})\omega^{\eta}\omega^{\zeta} +$$

$$+ c[J_i^{\xi\xi}(2\Omega_t^{\xi} + \Omega^{\zeta}\omega^{\eta}-\Omega^{\eta}\omega^{\zeta}) + (J_i^{\zeta\zeta}-J_i^{\eta\eta})(\Omega^{\zeta}\omega^{\eta}+\Omega^{\eta}\omega^{\zeta})] \qquad (II.101)$$

$$\frac{dH_c^{\eta}}{dt} = c^2[J_i^{\eta\eta}\Omega_s^{\eta} + (J_i^{\xi\xi}-J_i^{\zeta\zeta})\Omega^{\zeta}\Omega^{\xi}] +$$

$$J^{\eta\eta}\omega_t^{\eta} + (J^{\xi\xi}-J^{\zeta\zeta})\omega^{\zeta}\omega^{\xi} +$$

$$+ c[J_i^{\eta\eta}(2\Omega_t^{\eta} + \Omega^{\xi}\omega^{\zeta}-\Omega^{\zeta}\omega^{\xi}) + (J_i^{\xi\xi}-J_i^{\zeta\zeta})(\Omega^{\xi}\omega^{\zeta}+\Omega^{\zeta}\omega^{\xi})] \qquad (II.102)$$

II.13 ANALYSIS OF EQUILIBRIUM EQUATIONS IN THE LOCAL $\vec{\zeta}$, $\vec{\xi}$ AND $\vec{\eta}$ DIRECTIONS

Using (II.11), (II.14), (II.41), (II.92), (II.94), (II.96) and (II.100) to (II.102), the equations of equilibrium of forces and moments in the $\vec{\zeta}$, $\vec{\xi}$ and $\vec{\eta}$ directions can be written as

$$T_s - Q^{\xi}\Omega^{\eta} + Q^{\eta}\Omega^{\xi} - Wc_{12} + F_H^{\zeta}+\Delta^{\zeta} = ma^{\zeta} \tag{II.103}$$

$$Q_s^{\xi} - Q^{\eta}\Omega^{\zeta} + T\Omega^{\eta} - Wc_{22} + F_H^{\xi}+\Delta^{\xi} = ma^{\xi} + 2c\rho_i A_i \omega^{\eta} \tag{II.104}$$

$$Q_s^{\eta} - T\Omega^{\xi} + Q^{\xi}\Omega^{\zeta} - Wc_{32} + F_H^{\eta}+\Delta^{\eta} = ma^{\eta} - 2c\rho_i A_i \omega^{\xi} \tag{II.105}$$

$$(GI_e^p\Omega^{\zeta})_s + (EI_e^{\eta\eta}-EI_e^{\xi\xi})\Omega^{\eta}\Omega^{\xi} + M_H^{\zeta} + \Theta^{\zeta} =$$

$$J^{\zeta\zeta}\omega_t^{\zeta} + (J^{\eta\eta}-J^{\xi\xi})\omega^{\xi}\omega^{\eta} +$$

$$c[J_i^{\zeta\zeta}(2\Omega_t^{\zeta} + \Omega^{\eta}\omega^{\xi}-\Omega^{\xi}\omega^{\eta}) + (J_i^{\eta\eta}-J_i^{\xi\xi})(\Omega^{\eta}\omega^{\xi}+\Omega^{\xi}\omega^{\eta})] \tag{II.106}$$

$$(EI_e^{\xi\xi}\Omega^{\xi})_s + (GI_e^p - EI_e^{\eta\eta})\Omega^{\zeta}\Omega^{\eta} - Q^{\eta} + \Theta^{\xi} =$$

$$J^{\xi\xi}\omega_t^{\xi} + (J^{\zeta\zeta} - J^{\eta\eta})\omega^{\eta}\omega^{\zeta} +$$

$$c[J_i^{\xi\xi}(2\Omega_t^{\xi} + \Omega^{\zeta}\omega^{\eta}-\Omega^{\eta}\omega^{\zeta}) + (J_i^{\zeta\zeta}-J_i^{\eta\eta})(\Omega^{\zeta}\omega^{\eta}+\Omega^{\eta}\omega^{\zeta})] \tag{II.107}$$

$$(EI_e^{\eta\eta}\Omega^{\eta})_s - (GI_e^p - EI_e^{\xi\xi})\Omega^{\xi}\Omega^{\zeta} + Q^{\xi} + \Theta^{\eta} =$$

$$J^{\eta\eta}\omega_t^{\eta} + (J^{\xi\xi}-J^{\zeta\zeta})\omega^{\zeta}\omega^{\xi} +$$

$$c[J_i^{\eta\eta}(2\Omega_t^{\eta} + \Omega^{\xi}\omega^{\zeta}-\Omega^{\zeta}\omega^{\xi}) + (J_i^{\xi\xi}-J_i^{\zeta\zeta})(\Omega^{\xi}\omega^{\zeta}+\Omega^{\zeta}\omega^{\xi})] \tag{II.108}$$

where the effective torsional and bending rigidities are defined by

$$GI_e^p = GI^p - c^2 J_i^{\zeta\zeta} \tag{II.109}$$

$$EI_e^{\xi\xi} = EI^{\xi\xi} - c^2 J_i^{\xi\xi} \tag{II.110}$$

$$EI_e^{\eta\eta} = EI^{\eta\eta} - c^2 J_i^{\eta\eta} \tag{II.111}$$

In the derivation of (II.106) to (II.108) the internal cross sectional area of the riser has been assumed to be constant with s as in Sections (II.10) and (II.11). For compliant risers the differences between GI_e^p and GI^p, $EI_e^{\xi\xi}$ and $EI^{\xi\xi}$, and $EI_e^{\eta\eta}$ and $EI^{\eta\eta}$ are negligible. For typical configurations, e.g., de Oliveira and Morton [7], the difference is of order $\rho_i c^2/E \ll 1$. Similarly the right-hand sides of equations (II.106) to (II.108) are expected to be small for most practical circumstances (low frequencies). These terms are commonly neglected in the simpler Euler beam theory, see Crandall, et al. [18], Nordgren [11,13] and Garrett [14]. In the case of compliant risers subjected to rotating and reversing currents, the effects of the first two terms of the right-hand side of equation (II.106), which model torsional inertia, need to be investigated.

II.14 GOVERNING EQUATIONS AS A FIRST ORDER SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

It is convenient to convert the governing equations to a first order system of partial differential equations of the following symbolic form:

$$\vec{w}_s = \vec{f}(s,t,\vec{w}) + A(s)\cdot\vec{w}_t \tag{II.112}$$

where

$$\vec{w}(s,t) = [w_1(s,t),\ w_2(s,t),\ \ldots,\ w_N(s,t)]^T \qquad \text{(II.113)}$$

is the solution vector;

$$\vec{f}(s,t,\vec{w}) = [f_1(s,t,\vec{w}), f_2(s,t,\vec{w}),\ \ldots, f_N(s,t,\vec{w})]^T \qquad \text{(II.114)}$$

a given (nonlinear) vector function of s, t and $\vec{w}$; A(s) a NxN matrix, with elements which are functions of s.

It is further convenient to choose

$$\vec{w} = [T, Q^{\xi}, Q^{\eta}; \Omega^{\zeta}, \Omega^{\xi}, \Omega^{\eta}; \phi, \theta, \psi; x, y, z;$$
$$v^{\zeta}, v^{\xi}, v^{\eta}; \omega^{\zeta}, \omega^{\xi}, \omega^{\eta}; s^*]^T \qquad \text{(II.115)}$$

as our solution vector with N=19 unknown scalar variables. Nineteen independent equations are needed for a complete formulation of problem. These equations are enumerated below.

1. Three force equations (II.103) to (II.105), where
   - equations (II.15) are used to replace c_{i2}, i=1,2,3 in terms of ϕ, θ, ψ;
   - equations (II.33) to (II.35) are used to replace the acceleration components α^{ζ}, α^{ξ} and α^{η} in terms of $v^{\zeta}, v^{\xi}, v^{\eta}$ and $\omega^{\zeta}, \omega^{\xi}$ and ω^{η};

- the components of the external force per unit length are considered functions of s, t, a^{ζ}, a^{ξ}, a^{η}, θ,ϕ,ψ, x,y,z, v^{ζ}, v^{ξ}, v^{η}, ω^{ζ}, ω^{ξ}, ω^{η};
- the components of the structural damping force per unit length $\vec{\Delta}$ are considered functions of s, t and velocity components.

2. Three moment equations (II.106) to (II.108) where
   - M_H^{ζ}, the external moment per unit length around $\vec{\zeta}$ is considered function of s, t, ϕ,θ,ψ, x,y,z, $v^{\zeta},v^{\xi},v^{\eta}$, ω^{ζ}, ω^{ξ}, ω^{η};
   - the components of the structural damping moment per unit length, $\vec{\Theta}$, are considered functions of s, t and the angular velocity components.
3. Three equations (II.48) to (II.50) relating the spatial derivatives of the Euler angles with the Euler angles and the components of $\vec{\Omega}$.
4. Three equations (II.25) to (II.27) relating the spatial derivatives of the Cartesian coordinates x, y and z with Euler angles where e is replaced by T/EA.
5. Three compatibility relations (II.45) to (II.47), where we replace e by T/EA using equation (II.13).
6. Three relations (II.54) to (II.56) relating the components of $\vec{\Omega},\vec{\omega}$ and their time and spatial derivatives respectively; and
7. The following equation

$$s_s^* = 1 + T/EA \qquad \text{(II.116)}$$

resulting by introducing (II.13) in (II.23) to eliminate e. For the range of strains, e, analyzed in this work, the difference between s* and s is small, see assumption 3.

Estimates of the static response can be obtained by setting the components of $\vec{v}$ and $\vec{\omega}$ equal to zero in the governing equations, and replacing the external loads with their mean values. These mean values may, however, strongly depend upon the dynamic response, such as in the case of vortex induced dynamic lift, see Patrikalakis and Chryssostomidis [34,35]. The solution vector (II.115) for static calculations reduces to

$$\vec{w}_o=[T_o,Q_o^{\xi},Q_o^{\eta};\Omega_o^{\zeta},\Omega_o^{\xi},\Omega_o^{\eta};\phi_o,\theta_o,\psi_o;x_o,y_o,z_o;s_o^*]^T \qquad \text{(II.117)}$$

with N_o=13 unknown scalar variables, where subscript o denotes static quantities. To simplify the notation, subscript o has been omitted in the superscripts ζ, ξ and η appearing in (II.117). In the static case the governing equations symbolically reduce to

$$\vec{w}_{os} = \vec{f}_o(s,\vec{w}_o) \qquad \text{(II.118)}$$

where $\vec{f}_o$ is a given (nonlinear) vector function of s and $\vec{w}_o$ with N_o=13 scalar components. This vector function $\vec{f}_o$ includes the set of equations 1,2,3,4 and 7 defined above for the dynamic problem with the appropriate reduction of terms to indicate static response.

Efficient numerical solutions of the static problem for compliant risers can be found in Chryssostomidis and Patrikalakis [36] for a planar buoyant riser configuration without torsion. Efficient numerical solutions of the general static problem in three dimensions with spacing varying torsion can be found in Patrikalakis and Chryssostomidis [37].

Once the static response is determined, the linear dynamic equation of compliant risers can be obtained by subtracting the nonlinear static equations from the nonlinear dynamic equations in their vector form and linearizing for small dynamic motions and angles around the static configuration. The derivation of the linear dynamic equations for compliant risers and their solution using a novel combination of asymptotic and embedding techniques can be found in Patrikalakis and Chryssostomidis [38].

The solution of the complete nonlinear dynamic problem for compliant risers is a subject of current research. The prediction of the external loads $\vec{F}_H$ and $\vec{M}_H$ is one of the more important factors in a successful modeling of the static and dynamic behavior of compliant risers. Until rational methods allow the prediction of these loads in separated flows, approximate estimates based on strip theory and experimental 2-D flow models may be used for design purposes, see Patrikalakis [12] and Patrikalakis and Chryssostomidis [34,35].

II.15 BOUNDARY CONDITIONS

In the case of the static problem $N_0=13$ boundary conditions are necessary to complete equation (II.118). For the case of a Chinese Lantern configuration, de Oliveira and Morton [7] and de Oliveira, et al. [8], an approximate set of boundary conditions for the static problem involves prescription of ϕ,θ,ψ, x,y and z at s=0 and s=L and $s^*(0)=0$. For the case of a catenary configuration, Panicker and Yancey [6], the above boundary conditions at s=0 need to be modified to also express equilibrium of interaction forces and moments and kinematic compatibility with the lower rigid riser section.

For the case of a Chinese Lantern, an appropriate set of boundary conditions for the dynamic problem involves prescription of ϕ,θ,ψ, x,y and z at s=0 and s=L as functions of time and $s^*(0,t)=0$ for $t>0$. This gives 13 boundary conditions. The remaining six boundary conditions can be obtained by evaluating the angular velocities at s=0 and s=L using (II.37) to (II.39) and the prescribed values of ϕ,θ,ψ at each end as a function of time. Alternatively, the remaining six boundary conditions can be obtained by evaluating x_t,y_t,z_t at both ends, or by evaluating $v^{\zeta},v^{\xi},v^{\eta}$ at both ends. These velocity components can be obtained by

$$[v^{\zeta},v^{\xi},v^{\eta}]^T = C\cdot[x_t,y_t,z_t]^T \qquad \text{(II.119)}$$

where the elements of matrix C are given by (II.15). For the catenary configuration, the boundary conditions at s=L for the dynamic problem remain unaltered, while the boundary conditions at s=0 need to be modified to also express the equilibrium of interaction forces and moments, and kinematic compatibility with the lower rigid riser.

II.16 INITIAL CONDITIONS

An appropriate set of initial conditions involves the prescription of Euler angles ϕ,θ and ψ, their first partial derivatives with respect to time at t=0 and the function $s^*(s,0)$ for all s. The prescription of ϕ, θ, ψ, ϕ_t, θ_t and ψ_t for t=0 and all s allows the computation of all remaining variables comprising the solution vector at time t=0.

II.17 GOVERNING EQUATIONS FOR PLANAR RESPONSE

The governing equations are simplified significantly when motions are planar in the absence of torsion. We assume that the plane of motion is the vertical plane $O\vec{i}\vec{j}$, which is assumed to coincide with plane $\vec{\zeta}\vec{\xi}$ for all points along the riser. In this case, the solution vector (II.115) involves only ten non-trivial components and reduces to

$$\vec{w}_{2D} = [T, Q^{\xi}; \Omega^{\eta}; \phi; x, y; v^{\zeta}, v^{\xi}; \omega^{\eta}; s^*]^T \tag{II.120}$$

The remaining nine components are by assumption zero. The resulting ten governing equations are:

$$T_s - Q^{\xi}\Omega^{\eta} - Wc_{12} + F_H^{\zeta} + \Delta^{\zeta} = m(v_t^{\zeta} - \omega^{\eta} v^{\xi}) \tag{II.121}$$

$$Q_s^{\xi} + T\Omega^{\eta} - Wc_{22} + F_H^{\xi} + \Delta^{\xi} = m(v_t^{\xi} + \omega^{\eta} v^{\zeta}) + 2c\rho_i A_i \omega^{\eta} \tag{II.122}$$

$$(EI_e^{\eta\eta}\Omega^{\eta})_s + Q^{\xi} + \Theta^{\eta} = J^{\eta\eta}\omega_t^{\eta} + 2cJ_i^{\eta\eta}\Omega_t^{\eta} \tag{II.123}$$

$$\phi_s = \Omega^{\eta} \tag{II.124}$$

$$x_s = (1+T/EA)\cos\phi \tag{II.125}$$

$$y_s = (1+T/EA)\sin\phi \tag{II.126}$$

$$v^{\zeta}_s - \Omega^{\eta} v^{\xi} = T_t/EA \tag{II.127}$$

$$v^{\xi}_s + \Omega^{\eta} v^{\zeta} = (1+T/EA)\omega^{\eta} \tag{II.128}$$

$$\omega^{\eta}_s = \Omega^{\eta}_t \tag{II.129}$$

$$s^*_s = 1+T/EA \tag{II.130}$$

where

$$c_{12} = \sin\phi \tag{II.131}$$

$$c_{22} = \cos\phi \tag{II.132}$$

In the two-dimensional case, we observe that (II.124) and (II.129) or (II.39) lead to

$$\omega^{\eta} = \phi_t \tag{II.133}$$

The boundary and initial conditions appropriate for the two-dimensional problem are obtained from Sections (II.15) and (II.16) by eliminating the variables which are identically zero.

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Appendix A

BENDING, EXTENSIONAL AND TORSIONAL RIGIDITY OF A CROSS SECTION

Let us consider an arbitrary cross section of a compliant riser composed from n materials referred to a Cartesian system OXY, as in the Figure A-1 below:

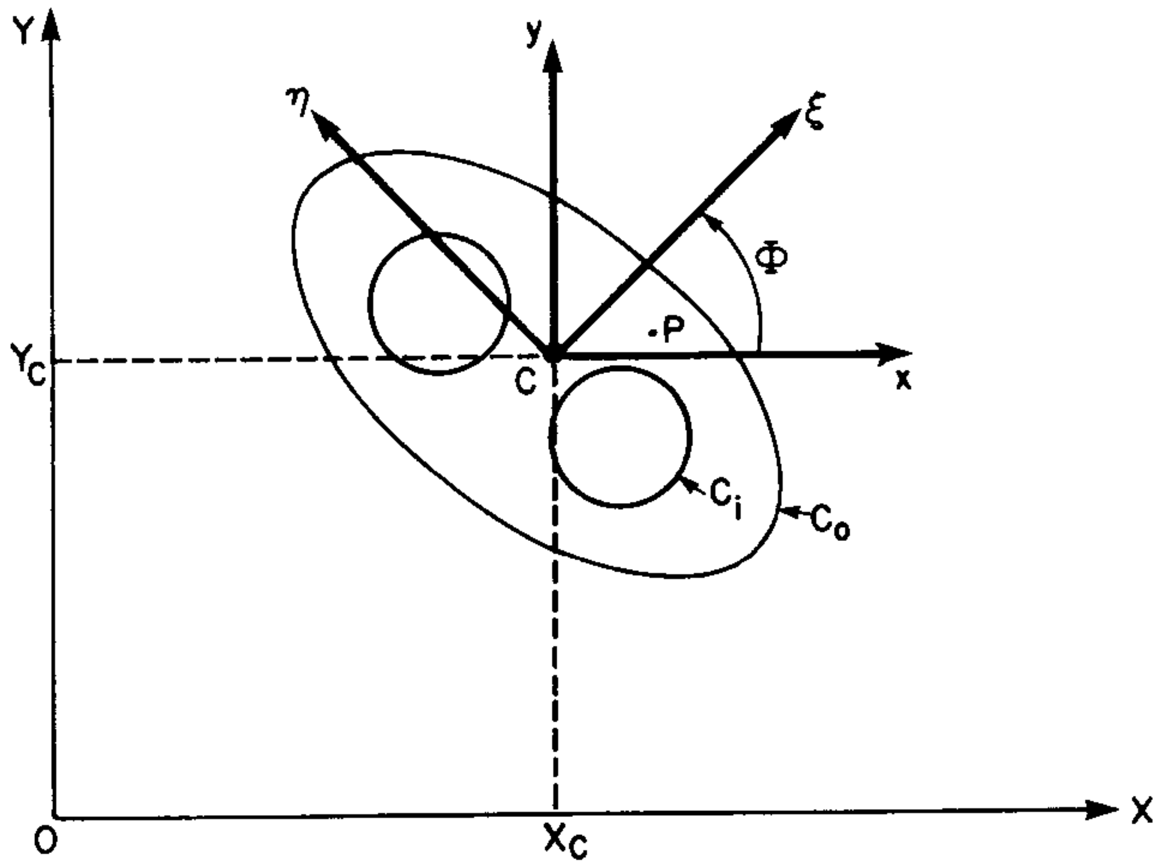

Figure A.1: Compliant Riser Cross-section

The cross section is bounded by an outer contour C_0 and by inner contours $C_i (i=1, \ldots M)$. Let P be an arbitrary point on a material of a cross section fully participating in bending and let E(P) the Young's modulus of the material at point P. Following Crandall, et al. [19], p. 424, it is convenient to define the "centroid", C, with a weighing function equal to E(P), i.e.:

$$X_C = \int_S dS\ E(P)X / \int_S dS\ E(P) \tag{A.1}$$

$$Y_C = \int_S dS\ E(P)Y / \int_S dS\ E(P) \tag{A.2}$$

where X_c, Y_c are the coordinates of C and S the surface of the materials of the cross section participating in bending.

Let us introduce a Cartesian system Cxy with axes Cx and Cy having the same direction as OX and OY and a system $C\xi\eta$ so that the angle between $\vec{Cx}$ and $\vec{C\xi}$ is Φ. Using $x = X-X_c$, $y = Y-Y_c$ and

$$x = \xi \cos \Phi - \eta \sin \Phi \tag{A.3}$$

$$y = \xi \sin \Phi + \eta \cos \Phi \tag{A.4}$$

together with (A.1) and (A.2), we find that

$$\int_S E(P)\ \xi\ dS = \int_S E(P)\ \eta\ dS \tag{A.5}$$

The bending rigidities of the cross section about x and y and the cross product of bending rigidity are defined by

$$EI^{xx} = \int_S E(P)y^2 dS \tag{A.6}$$

$$EI^{yy} = \int_S E(P)x^2 dS \tag{A.7}$$

$$EI^{xy} = \int_S E(P)xy dS \tag{A.8}$$

Considering an elementary area dS of material with coordinates

$$\xi = x \cos \Phi + y \sin \Phi \tag{A.3a}$$

$$\eta = -x \sin \Phi + y \cos \Phi \tag{A.4a}$$

and letting,

$$EI^{\xi\xi} = \int_S E(P)\, \eta^2 dS \tag{A.9}$$

$$EI^{\eta\eta} = \int_S E(P)\xi^2 dS \tag{A.10}$$

$$EI^{\xi\eta} = \int_S E(P)\xi\eta dS \tag{A.11}$$

we find following Timoshenko and Young [39], p. 354:

$$EI^{\xi\xi} = \tfrac{1}{2}(EI^{xx}+EI^{yy}) + \tfrac{1}{2}(EI^{xx}-EI^{yy})\cos 2\Phi -$$

$$- EI^{xy}\sin 2\Phi \tag{A.12}$$

$$EI^{\eta\eta} = \frac{1}{2}(EI^{xx}+EI^{yy}) - \frac{1}{2}(EI^{xx}-EI^{yy})\cos 2\Phi +$$

$$+ EI_{xy} \sin 2\Phi \qquad (A.13)$$

$$EI^{\xi\eta} = \frac{1}{2}(EI^{xx}-EI^{yy})\sin 2\Phi + EI^{xy}\cos 2\Phi \qquad (A.14)$$

The value of Φ can be chosen so that $EI^{\xi\xi}$ is, for example, maximum. This requires $EI^{\xi\xi}_{\Phi} = 0$ and $EI^{\xi\xi}_{\Phi\Phi} < 0$. These relations give

$$\tan 2\Phi = \frac{2\ EI^{xy}}{EI^{yy}-EI^{xx}},\ \sin 2\Phi < 0 \qquad (A.15)$$

If these relations are valid, then $EI^{\eta\eta}_{\Phi} = 0$ and $EI^{\eta\eta}_{\Phi\Phi} > 0$ and therefore $EI^{\eta\eta}$ is simultaneously minimum. The value of Φ determined by (A.15) leads to

$$EI^{\xi\eta} = 0 \qquad (A.16)$$

The axes $C\xi$ and $C\eta$ corresponding to this value of Φ are called centroidal principal axes and because of (A.5) and (A.16) simplify the constitutive relations for the rod in bending. When (A.15) is valid, we obtain

$$EI^{\xi\xi} = \frac{1}{2}(EI^{xx}+EI^{yy}) + \left[\left(\frac{EI^{xx}-EI^{yy}}{2}\right)^2+(EI^{xy})^2\right]^{1/2} \qquad (A.17)$$

$$EI^{\eta\eta} = \frac{1}{2}(EI^{xx}+EI^{yy}) - \left[\left(\frac{EI^{xx}-EI^{yy}}{2}\right)^2+(EI^{xy})^2\right]^{1/2} \qquad (A.18)$$

The definition of the extensional rigidity of the cross section is

$$EA = \int_{S'} E(P)dS \qquad (A.19)$$

where S' is the surface of the material resisting extension.

Finally, the torsional rigidity GI^P of a cross section can be determined by the stress function method, as described, for example, in Love [9], Landau and Lifshitz [10] and Timoshenko and Goodier [40]. For complicated sections, approximate expressions for the stress function can be obtained from an energy method, Timoshenko and Goodier [40].

All above derivations assume that the various materials composing the cross section are uniform and work perfectly together in the corresponding deformation mode; i.e., that the corresponding deformations are continuous across surfaces of materials taken into account. When the materials are not uniform (as for example in the case of steel armor wires protecting pipes made from synthetic materials), more complicated analysis and, often, experiments are necessary, Timoshenko [41], de Oliveira and Morton [7] and de Oliveira, et al. [8]. Finally, the above derivations do not account for changes of the rigidities as a function of the level of deformation.

Appendix B

DERIVATION OF CONSTITUTIVE RELATIONS IN THE $C\vec{t}\vec{n}\vec{b}$ SYSTEM

Let us assume that the centerline of the rod is a continuous curve composed of the centroids, C, of all cross sections, where C is defined in Appendix A. The local tangential, normal and binormal vectors of the centerline at each C can be defined in terms of the position vector $\vec{R}$ of C with respect to an inertial system OXYZ. We choose OXYZ so that OY is vertical and positive upwards. If, for example, the lower end of the rod is fixed with respect to the earth, then we may, for convenience, choose O to coincide with the centroid of the lower end cross section of the rod. Otherwise, we may choose O to coincide with a particular convenient point fixed with respect to the earth. Point C can be identified in terms of $s^*(s,t)$, the stretched arc length from O or in terms of s the unstretched arc length from O. In the subsequent analysis differentiations with respect to s^* to determine curvatures and torsion will be replaced by differentiations with respect to s because the extensional strain $e=s^*_s-1$ is very small compared to one, Love [9].

Let U be the column triad of unit vectors along OX, OY and OZ:

$$U = [\vec{i}\ \vec{j}\ \vec{k}]^T \tag{B.1}$$

where $[\ ]^T$ denotes transpose. We may than analyze $\vec{R}$, the vector $\vec{OC}$, as

$$\vec{R} = (x^1, x^2, x^3)\cdot U \tag{B.2}$$

The tangential unit vector, $\vec{t}$, to the centerline at C is defined by

$$\vec{t} = \vec{R}_{s*} = \frac{\vec{R}_s}{1+e} = (\alpha^1, \alpha^2, \alpha^3) \cdot U = \vec{R}_s \tag{B.3}$$

see Eisenhart [26], where subscript s denotes partial derivative with respect to s, the unstretched arc length of the centerline from the lower end of the rod, where $\alpha^i = x^i_{s*}$ for i = 1,2,3. Since $\vec{t}$ is a unit vector, we obtain

$$e = (\vec{R}_s \cdot \vec{R}_s)^{1/2} - 1$$

The normal unit vector, $\vec{n}$, to the centerline at C is defined by

$$\vec{n} = K^{-1}\vec{R}_{ss} = (\beta^1, \beta^2, \beta^3) \cdot U \tag{B.4}$$

where $\beta^i = K^{-1}x^i_{ss}$ for i = 1,2,3 and K is the curvature of the centerline at C, defined by

$$K = |\vec{R}_{ss}| \tag{B.5}$$

where $| \quad |$ denotes the length of a vector, Eisenhart [26].

The binormal unit vector, $\vec{b}$, to the centerline at C is defined so that $\vec{n}$, $\vec{b}$ and $\vec{t}$ form a right-handed system:

$$\vec{b} = \vec{t} \times \vec{n} = (\gamma^1, \gamma^2, \gamma^3) \cdot U \tag{B.6}$$

where $\gamma^i = K^{-1}(x^j_s x^k_{ss} - x^k_s x^j_{ss})$ and the indices i,j,k take the values 1,2,3 cyclically, Eisenhart [26].

The rates of change of $\vec{t}$, $\vec{n}$ and $\vec{b}$ with respect to s can be shown to obey the following identities called Frenet relations:

$$\vec{t}_s = K\vec{n} \tag{B.7}$$

$$\vec{n}_s = \tau\vec{b} - K\vec{t} \tag{B.8}$$

$$\vec{b}_s = -\tau\vec{n} \tag{B.9}$$

where τ is the geometric torsion, Eisenhart [26], or measure of tortuosity, Love [9] of the centerline at C. Note that Love [9] uses the symbol $1/\Sigma$ instead of τ.

The negative sign is used in equation (B.9) so that the torsion τ is positive when the vector triad $\vec{t},\vec{n},\vec{b}$ rotates in a right-handed sense about $\vec{t}$ as it progresses along the curve, Hildebrand [42]. It can be shown that τ is given by the following equation:

$$\tau = K^{-2}\, \vec{R}_s \cdot (\vec{R}_{ss} x \vec{R}_{sss})$$

or equivalently

$$\tau = K^{-2} \det \begin{bmatrix} x_s^1 & x_s^2 & x_s^3 \\ x_{ss}^1 & x_{ss}^2 & x_{ss}^3 \\ x_{sss}^1 & x_{sss}^2 & x_{sss}^3 \end{bmatrix} \tag{B.10}$$

where det [.] denotes the determinant of a matrix, Eisenhart [26]. From the definition of τ it follows that τ is zero for a plane curve, Eisenhart [26].

For a fixed time, the change of vector $\vec{t}$ between two neighboring points of the centerline is $d\vec{t} = \vec{d\Phi} x \vec{t}$, where $\vec{d\Phi}$ is the vector of the angle of infinitesimal rotation of the system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ at s+ds relative to the system $C\vec{\zeta}\vec{\xi}\vec{\eta}$ at s. Therefore, dividing by ds and using equation (II.10) we find that

$$\vec{t}_s = \vec{\Omega} \times \vec{t} \tag{B.11}$$

Using equations (B.11), (B.6) and (B.7) we find that

$$\vec{t} \times (\vec{\Omega} x \vec{t}) = (\vec{t} \cdot \vec{t})\vec{\Omega} - (\vec{t} \cdot \vec{\Omega})\vec{t} = \vec{t} x \vec{t}_s = K\vec{t} x \vec{n} = K\vec{b}$$

and therefore, since $\quad \Omega^\zeta = \vec{\zeta} \cdot \vec{\Omega} = \vec{t} \cdot \vec{\Omega}$

$$\vec{\Omega} = K\vec{b} + \Omega^\zeta \vec{t} \tag{B.12}$$

Let f be the angle between $\vec{n}$ and $\vec{\eta}$, as in Figure B.1, Love [9], and ℓ_1, ℓ_2, ℓ_3 the direction cosines of $\vec{b}$ with respect to $C\vec{\zeta}\vec{\xi}\vec{\eta}$. Using (B.9) we find that

$$\tau^2 = \ell_{1_s}^2 + \ell_{2_s}^2 + \ell_{3_s}^2 \tag{B.13}$$

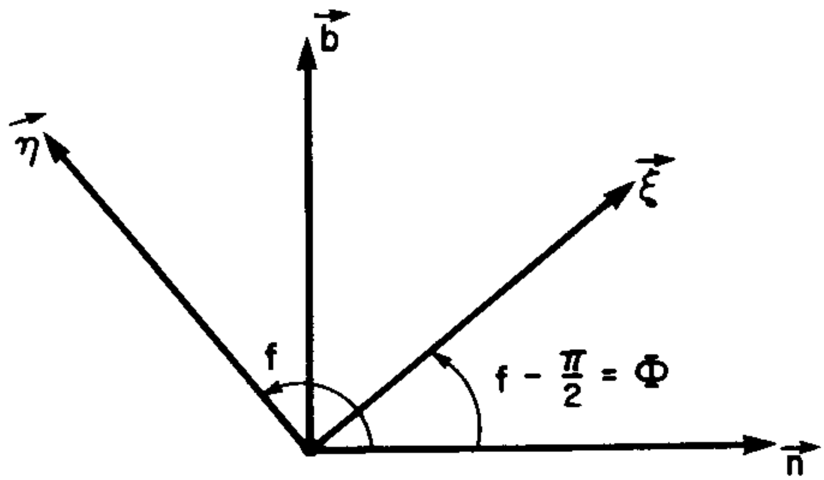

Figure B.1: Coordinate System for a Cross-section

where, of course, $\ell_1 = 0$, because $\vec{b}\cdot\vec{\zeta} = 0$ and as in equations (B.1) to (B.11) subscript s denotes the total derivative with respect to s for a fixed time t.

Using the decompositions of $\vec{\Omega} = [\Omega^\zeta, \Omega^\xi, \Omega^\eta]\cdot U''$ and of equation (B.12) we find that

$$\ell_2 = \Omega^\xi K^{-1} = -\cos f \tag{B.14}$$

$$\ell_3 = \Omega^\eta K^{-1} = \sin f \tag{B.15}$$

with

$$K^2 = \Omega^{\xi 2} + \Omega^{\eta 2} \tag{B.16}$$

and

$$\tan f = -\Omega^{\eta}/\Omega^{\xi} \tag{B.17}$$

see Love [9].

Using the fact that

$$\vec{b}_s = (\vec{b}_s)_{\vec{C}\vec{\zeta}\vec{\xi}\vec{\eta}} + \vec{\Omega}\times\vec{b} \tag{B.18}$$

see Landau and Lifshitz [10] and Crandall, et al. [18], or explicitly

$\ell_{2s} = \sin f\ f_s - \Omega^{\zeta}\sin f$ and $\ell_{3s} = \cos f\ f_s - \Omega^{\zeta}\cos f$

and substituting in equation (B.13), we find that:

$$\Omega^{\zeta} = f_s + \tau \tag{B.19}$$

where the convention that $\vec{\zeta}\vec{\xi}\vec{\eta}$ and $\vec{t}\vec{n}\vec{b}$ are right-handed systems is used, see Love [9]. Alternatively equation (B.19) can be obtained by using $\vec{b} = \ell_2\vec{\xi}+\ell_3\vec{\eta}$ and noting that the left-hand side of equation (B.18) is equal to $-\tau\vec{n}$ and the right-hand side is equal to $f_s(\sin f\vec{\xi}+\cos f\vec{\eta})-\vec{\xi}\sin f\Omega^{\zeta}-\vec{\eta}\cos f\Omega^{\zeta}+\vec{\zeta}(\Omega^{\xi}\sin f+\Omega^{\eta}\cos f)$ and multiplying both sides by $\vec{n}$.

We are now in a position to express the components of the bending moment along $\vec{b}$ and $\vec{n}$ in terms of its components along $\vec{\xi}$ and $\vec{\eta}$ by the relations:

$$M^b = -M^\xi \cos f + M^\eta \sin f \tag{B.20}$$

$$M^n = M^\xi \sin f + M^\eta \cos f \tag{B.21}$$

Using (II.11), (B.14) and (B.15) we obtain

$$M^b = K[EI^{\xi\xi}\cos^2 f + EI^{\eta\eta}\sin^2 f] \tag{B.22}$$

$$M^n = \frac{1}{2} K[EI^{\eta\eta} - EI^{\xi\xi}] \sin 2f \tag{B.23}$$

We now express $EI^{\xi\xi}$ and $EI^{\eta\eta}$ in terms of EI^{nn}, EI^{bb} and EI^{nb} using the relations (A.12) to (A.14) and (A.16), where $\Phi = f - \pi/2$ and x,y of Figure A-1 are replaced by $\hat{n}$, $\hat{b}$ of Figure B.1, respectively:

$$EI^{\xi\xi} = \frac{1}{2}(EI^{nn} + EI^{bb}) - \frac{1}{2}(EI^{nn} - EI^{bb})\cos 2f + EI^{nb}\sin 2f \tag{B.24}$$

$$EI^{\eta\eta} = \frac{1}{2}(EI^{nn} + EI^{bb}) + \frac{1}{2}(EI^{nn} - EI^{bb})\cos 2f - EI^{nb}\sin 2f \tag{B.25}$$

$$EI^{nb} = \frac{1}{2}(EI^{bb} - EI^{nn})\tan 2f \tag{B.26}$$

Using (B.22) and (B.26) we therefore find that

$$M^b = EI^{bb} \cdot K \tag{B.27}$$

$$M^n = -EI^{nb} \cdot K \tag{B.28}$$

These equations imply that in rotationally uniform rods, where $EI^{nb}=0$, the bending moment is exclusively directed in the local $\hat{b}$ direction, Love [9]. The approximation $M^n=0$ was, for example, used in the modelling of conventional risers, Patrikalakis [12], Nordgren [11,13] and Garrett [14].

Appendix C

CONSTITUTIVE RELATION BETWEEN T and e

Introduction

For the case of a cable composed of a homogeneous, isotropic and linearly elastic material, Goodman and Breslin [28] showed that tension in the material P and small extensional strain of the centerline are related by:

$$P = EAe - 2\nu A_o p_o \tag{C.1}$$

where EA is the extensional rigidity of the cable, ν Poisson's ratio and p_o the water head. For a cable the effective tension is

$$T = P + p_o A_o \tag{C.2}$$

and therefore,

$$T = EA\ e + (1-2\nu) p_o A_o \tag{C.3}$$

Previous investigators, such as Goodman and Breslin [28], Triantafyllou [22], Triantafyllou and Bliek [23], Bliek [24] and Triantafyllou, et al. [25], used a Poisson's ratio $\nu = 1/2$ because of the simplification of the constitutive relation and subsequent analysis. The assumption $\nu = 1/2$ for a cable element is equivalent to zero volume expansion, Timoshenko and Goodier [40] and leads to:

$$T = EA\ e \tag{C.4}$$

Values of Poisson's ratio for engineering materials vary from 0.1 for concrete to 0.5 for rubber. For most metals it is between 0.25 and 0.35, see Harris and Crede [43].

Analysis for the Case of a Uniform Cylindrical Tube

We start with a straight unstretched differential element of the tube, ds, in air bounded by two adjacent cross-sections and with inner and outer radii equal to r_i and r_o, respectively. The tube is filled with a liquid of density ρ_i and an internal overpressure, p, above the atmospheric is applied. Finally the tube is immersed in a liquid of density ρ_w and is subjected to external loads. Under the action of all loads ds extends to ds*, so that the extensional strain of the centerline is

$$e = s_s^* - 1 \tag{C.5}$$

It is assumed that $e \ll 1$. In addition the centerline of the tube is no longer straight. In the sequel we perform an approximate analysis to determine the relation between tension, P, and small extensional strain, e, assuming that ds* is straight and additionally that:

1. The cross section remains circular under the action of all loads. This is very accurate if $2r_o/(r_o-r_1)$ is less than 25 to 30 and the curvature is not very large, von Karman [44].

2. The normal stresses parallel to $\vec{t}$ due to tension are constant throughout the material cross-section.

3. The radii r_o and r_i are small compared to the length of the tube so that the internal and external hydrostatic pressures are essentially constant for all points of each cross-section; i.e., the loading is axisymmetric.

4. Shear stresses are negligible.

5. Body (gravity) forces are neglected from the field equations of elastic stress equilibrium because their global effect is reflected in the value of the tension.

Using a local cylindrical coordinate system (r,θ,z) where r is the radius, θ the polar angle and z the tangential distance, we find that σ_r, σ_θ and σ_z are independent of θ and that we need to satisfy the following equation of equilibrium in the radial direction

$$\frac{\partial\sigma_r}{\partial r} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \qquad (C.6)$$

where

$$\sigma_r = (1-\nu^2)^{-1}[E(\varepsilon_r+\nu\varepsilon_\theta) + \nu(\nu+1)\sigma_z] \qquad (C.7)$$

$$\sigma_\theta = (1-\nu^2)^{-1}[E(\varepsilon_\theta+\nu\varepsilon_r) + \nu(\nu+1)\sigma_z] \qquad (C.8)$$

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r} \qquad (C.9)$$

and ε_r, ε_θ are the radial and circumferential strains and u the radial displacement. Using (C.6) to (C.9) and our assumptions we obtain

$$r^2 u_{rr} + r u_r - u = 0 \tag{C.10}$$

which has the following general solution

$$u = Ar + \frac{B}{r}$$

The boundary conditions are

$$\sigma_r = -p_i \text{ at } r=r_i \text{ and } \sigma_r = -p_o \text{ at } r=r_o \tag{C.11}$$

where $p_i = p + \rho_i g(h_i - y)$, $p_o = \rho_w g(h_w - y)$

Equations (C.7), (C.8) and (C.11) lead to

$$A = \frac{1-\nu}{E} \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} - \frac{\nu}{E} \sigma_z \tag{C.12}$$

$$B = \frac{1+\nu}{E} \frac{r_o^2 r_i^2}{r_o^2 - r_i^2} (p_i - p_o) \tag{C.13}$$

Using (C.7) to (C.9) and (C.12), (C.13) we find that

$$\sigma_r + \sigma_\theta = 2 \frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \tag{C.14}$$

which is independent of r and because of

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r+\sigma_\theta)\right] \tag{C.15}$$

the resulting extensional strain ε_z is uniform throughout the cross sectional (i.e., equal to e) and can be estimated from

$$\varepsilon_z = \frac{\sigma_z}{E} - \frac{2\nu}{E}\,\frac{p_i r_i^2 - p_o r_o^2}{r_o^2 - r_i^2} \tag{C.16}$$

where

$$\sigma_z = P/A \text{ and } A=\pi(r_o^2-r_i^2) \tag{C.17}$$

Using $A_o=\pi r_o^2$ and $A_i = \pi r_i^2$ we find that

$$P=EAe + 2\nu(p_iA_i-p_oA_o) \tag{C.18}$$

Using the definition of the effective tension T (II.91.2)

$$T = P + p_oA_o-p_iA_i \tag{C.19}$$

where the term $\rho_i c^2 A_i$ has been neglected because $\rho_i c^2 \ll p_i$, we obtain

$$T = EAe + (1-2\nu)(p_oA_o-p_iA_i) \tag{C.20}$$

which is the required constitutive relation between T and e.

For the reasons given in the Introduction, we will adopt the value $\nu = 1/2$, which leads to

$$T = EAe \tag{C.21}$$

We will continue to use (C.21) for multilayer tubes and multitube configurations where EA is taken equal to the overall extensional rigidity of the riser.